Winter 2018 Probability Qualifying Exam

December 6, 2017

Instructions:

1. Read all questions carefully. If you are confused ask me!
2. You should have 5 pages including this page. Make sure you have the right number of pages.
3. Unless otherwise noted all conventions and notation follow that of Çinlar. If you are confused about notation, ask!
4. If necessary you may use the back of pages.

Name:____________________________________

UO ID:____________________________________

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	10	10	10	10	10	10	10	10	90
Score:										

[10 pts] 1. Let $\mathcal{F} = \{\mathcal{F}_n : n \in \mathbb{N}\}$ be a filtration and set $\mathcal{F}_\infty = \bigvee \mathcal{F}_n$. Show that for each bounded random variable $V \in \mathcal{F}_\infty$ there is a sequence of bounded random variables $V_n \in \mathcal{F}_n$ such that V_n converges to V in L^1.

> **Solution:** Define the p-system $\mathcal{C} = \bigcup \mathcal{F}_n$. define
>
> $$\mathcal{M} = \{V \in \mathcal{F}_\infty : V \text{ bdd, and there exist bdd } V_n \in \mathcal{F}_n \text{ so that } V_n \to V \text{ in } L^1\}.$$
>
> Clearly $\mathcal{M}$ contains all indicators of sets in $\mathcal{C}$ and is a vector space. We need only show that if U_k are positive random variables in $\mathcal{M}$ which increase to some V, then $V \in \mathcal{M}$.
>
> For each U_k there exist bounded $U_{k,n} \in \mathcal{F}_n$ such that $\mathbb{E}|U_{k,n} - U_k| \to 0$. Set $n_0 = 0$ and define $n_k > n_{k-1}$ where
>
> $$|U_{k,n_k} - U_k| < \frac{1}{k}$$
>
> Set $\widehat{U}_k := U_{k,n_k}$. Then,
>
> $$\mathbb{E}|\widehat{U}_k - V| \le \mathbb{E}|U_k - V| + \mathbb{E}|U_k - \widehat{U}_k| \to 0$$
>
> Finally, set $V_0 = 0$ and $V_n = \widehat{U}_k$ for all integers in $(n_k, n_{k+1}]$. It follows that $V_n \to V$ in L^1 and each of the V_n are bounded in $\mathcal{F}_n$ by the filtration property. That is $V \in \mathcal{M}$.

[10 pts] 2. Let (X_n) be a sequence of random variables with positive, finite variance. Show that

$$\limsup_n \frac{X_1 + \cdots + X_n}{\sqrt{n}} = +\infty \quad \text{a.s.}$$

> **Solution:** Kolmogorov's 0-1 law on event $\limsup_n \frac{X_1+\cdots+X_n}{\sqrt{n}} > c$. The positive variance keeps the random variables from all being identically 0.

[10 pts] 3. Give an example of σ-algebras $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ (all sub σ-algebras of $\mathcal{H}$) so that $\mathcal{F}_1$ and $\mathcal{F}_2$ are independent, but not conditionally independent given $\mathcal{F}$. Justify.

> **Solution:** One possible solution: let X and Y be i.i.d. random variables and set $\mathcal{F}_1 = \sigma X$ and $\mathcal{F}_2 = \sigma Y$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are independent. Set $Z = X + Y$ and $\mathcal{F} = \sigma Z$, then $\mathcal{F}_1$ and $\mathcal{F}_2$ are not conditionally independent given $\mathcal{F}$. This can be demonstrated easily with, for instance, X and Y being i.i.d. Bernoulli random variables.

[10 pts] 4. Suppose (X_n) is a sequence of real-valued random variables which converges in probability. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function show that $f(X_n)$ also converges in probability.

Solution: Suppose (X_n) converges to X in probability. Then, for any subsequence N, $(X_n)_{n\in N}$ converges to X in probability, and there exists a subsequence N' of N along which X_n converges almost surely. Thus, by continuity of f, $(f(X_n))_{n\in N'}$ converges almost surely to $f(X)$. It follows that, since every subsequence of $f(X_n)$ contains a further subsequence which converges almost surely to $f(X)$, then $f(X_n)$ converges in probability to $f(X)$.

[10 pts] 5. Suppose $(X_n)_{n\in\mathbb{N}}$ is adapted to the filtration $\mathcal{F} = (\mathcal{F}_n)$, each X_n is integrable and define

$$\widehat{X}_n = \frac{1}{n+1}(X_0 + \cdots + X_n)$$

Show that if $\mathbb{E}_n[X_{n+1}] = \widehat{X}_n$, then $\widehat{X}_n$ is an $\mathcal{F}$-martingale.

Solution: Clearly each $\widehat{X}_n$ is adapted and integrable. Moreover,

$$\begin{aligned}
\mathbb{E}_n[\widehat{X}_{n+1}] &= \mathbb{E}_n\left[\frac{1}{n+2}(X_0 + \cdots + X_n + X_{n+1})\right] \\
&= \frac{1}{n+2}\left\{X_0 + \cdots + X_n + \mathbb{E}_n[X_{n+1}]\right\} \\
&= \frac{1}{n+2}\left\{X_0 + \cdots + X_n + \widehat{X}_n\right\} \\
&= \frac{1}{(n+2)(n+1)}\left\{(n+1)(X_0 + \cdots + X_n) + X_0 + \cdots + X_n\right\} \\
&= \frac{1}{(n+1)}\left\{(X_0 + \cdots + X_n)\right\} = \widehat{X}_n.
\end{aligned}$$

[10 pts] 6. Suppose $X = (X_n)$ is adapted and integrable. Show that there exists a martingale M and a predictable process A such that $M_0 = A_0 = 0$ and

$$X_n = X_0 + M_n + A_n.$$

Show that this decomposition is unique up to equivalence (that is M_n and A_n are unique almost surely).

> **Solution:** Set $M_0 = A_0 = 0$ and define A_n and M_n via increments
>
> $$A_{n+1} - A_n = \mathbb{E}_n[X_{n+1} - X_n] \qquad \text{and} \qquad M_{n+1} - M_n = (X_{n+1} - X_n) - (A_{n+1} - A_n)$$
>
> Clearly (M_n) is a martingale and (A_n) is predictable. For uniqueness, suppose $X_n = X_0 + M'_n + A'_n$ is another decomposition. Then, $B = A - A' = M' - M$ is both predictable and a martingale. That is,
>
> $$B_{n+1} - B_n = \mathbb{E}_n[B_{n+1} - B_n] = 0 \quad \text{almost surely.}$$
>
> Since $B_n = B_0 = 0$ we have $A_n = A'_n$ and $M_n = M'_n$ almost surely as desired.

[10 pts] 7. Suppose $X = (X_n)$ is a L^1-bounded martingale. Write $X_n = X_n^+ - X_n^-$ where X_n^+ and X_n^- are non-negative. Show that $X^+ = (X_n^+)$ is a submartingale and that

$$Y_n = \lim_m \mathbb{E}_n[X_{n+m}^+]$$

defines a positive L^1-bounded martingale Y.

> **Solution:** The first part is simply Jensen's inequality with $f(x) = 0 \vee x$ for conditional expectation.
>
> Since (X_n^+) is L^1-bounded, it is uniformly integrable, and hence converges. That is there exists an L^1 random variable X_∞^+ such that
>
> $$\lim_n X_n^+ = X_\infty^+ \quad \text{a.s.}$$
>
> and (X_n^+) extends to a submartingale on $\mathbb{N} \cup \{+\infty\}$.
>
> Thus,
>
> $$\mathbb{E}_n[Y_{n+1}] = \mathbb{E}_n\left[\lim_m \mathbb{E}_{n+1}[X_{n+m+1}^+]\right] = \mathbb{E}_n\left[\mathbb{E}_{n+1}[\lim_m X_{n+m+1}^+]\right]$$
>
> where we have used the submartingale condition to justify the use of monotone convergence for conditional expectation. Another use of monotone convergence then implies,
>
> $$\mathbb{E}_n[Y_{n+1}] = \mathbb{E}_n\left[\mathbb{E}_{n+1}[X_\infty^+]\right] = \mathbb{E}_n[X_\infty^+] = \mathbb{E}_n\left[\lim_m \mathbb{E}_n[X_{n+m}^+]\right] = \mathbb{E}_n[Y_n]$$
>
> as desired.

[10 pts] 8. Fix $s > 1$. Suppose $\{N_p : p \text{ prime}\}$ is an independency of non-negative integer valued random variables with $\mathbb{P}\{N_p = n\} = \frac{p^{-ns}}{(1-p^{-s})}$. Show that

$$\prod_p p^{N_p}$$

is almost surely finite.

Solution: This is equivalent to there being only finitely many p with $N_p > 0$ almost surely. We use Borel-Cantelli

$$\mathbb{P}\{N_p > 0\} = \frac{1}{1-p^{-s}} - 1 = \frac{p^{-s}}{1-p^{-s}}.$$

If we set $C = 1 - 2^{-s}$ then,

$$\mathbb{P}\{N_p > 0\} \leq \frac{p^{-s}}{C},$$

and

$$\sum_p \mathbb{P}\{N_p > 0\} \leq \frac{1}{C} \sum_p p^{-s} < \infty$$

[10 pts] 9. Let X be a real-valued random variable with mean μ and finite variance σ^2. Show that

$$\mathbb{P}\{X - \mu \geq x) \leq \frac{\sigma^2}{\sigma^2 + x^2}$$

Solution: Let $Y = X - \mu$ and let u be any real number. Then Chebyshev's inequality implies

$$\mathbb{P}\{Y > x\} = \mathbb{P}\{(Y+u)^2 > (x+u)^2\} \leq \frac{\mathbb{E}[(Y+u)^2]}{(x+u)^2} = \frac{\sigma^2 + u^2}{(x+u)^2}$$

We see the right hand side is maximized when $u = \sigma^2/x$ and the inequality follows.

End of Exam