ALGEBRA QUALIFYING EXAM, WINTER 2017.
<u>SOLUTIONS</u>

Your Name:

Conventions: all rings and algebras are assumed to be unital.

Part I. True or false? If true provide a brief explanation, if false provide a counterexample (10 points each):

1. Let $\mathbf{A}, \mathbf{B}$ be categories and A_1, A_2 be objects of $\mathbf{A}$. If $\mathcal{F}, \mathcal{G} : \mathbf{A} \to \mathbf{B}$ are isomorphic functors and $f, g \in \mathrm{Hom}_{\mathbf{A}}(A_1, A_2)$, then $\mathcal{F}(f) = \mathcal{F}(g)$ if and only if $\mathcal{G}(f) = \mathcal{G}(g)$.

   True. Let α be an isomorphism from $\mathcal{F}$ to $\mathcal{G}$. Then $\mathcal{G}f = \alpha_{A_2} \circ \mathcal{F}f \circ \alpha_{A_2}^{-1}$ and $\mathcal{G}g = \alpha_{A_2} \circ \mathcal{F}g \circ \alpha_{A_2}^{-1}$, so $\mathcal{F}(f) = \mathcal{F}(g)$ implies $\mathcal{G}(f) = \mathcal{G}(g)$, and the converse is similar.

2. If A is a finite dimensional commutative algebra over a field then all irreducible A-modules are 1-dimensional.

False. $\mathbb{Q}C_3$ has 2-dimensional irreducible module. Or regular module over the $\mathbb{R}$-algebra $\mathbb{C}$.

3. Let $A \supseteq R$ be an integral ring extension, and assume that A is a domain. If every non-zero prime ideal of R is a maximal ideal, then every non-zero prime ideal of A is also maximal.

True. By the Incomparability Theorem, the only prime ideal of A lying over 0 is 0. Now it follows from the Maximality Theorem that if P is a non-zero prime ideal of A, then P is maximal, since $P \cap R$ is non-zero prime.

4. If β is algebraic over $F(\alpha)$ and β is transcendental over F then α is algebraic over $F(\beta)$.

True. By the Main Criterion, if α was transcendental over $F(\beta)$, then $\{\alpha, \beta\}$ would be algebraically independent. But then β would be transcendental over $F(\alpha)$.

Second solution: There is a polynomial $f \in F(\alpha)[x]$ with $f(\beta) = 0$. Any $a \in F(\alpha)$ can be written as $g(\alpha)/h(\alpha)$. Clearing denominators from the coefficients in f, we get a polynomial $g \in F[x, y]$ such that $f(\beta, \alpha) = 0$. This gives a polynomial $g \in F[\beta][x]$ with $g(\alpha) = 0$. Furthermore, $g \neq 0$ since the coefficients cannot all be zero because β is transcendental over F.

5. $\mathbb{Q}[x, x^{-1}]$ is a projective $\mathbb{Q}[x]$-module.

False. Consider $\mathbb{Q}[x] \to \mathbb{Q}$ which is evaluation at 1, and $\mathbb{Q}[x, x^{-1}] \to \mathbb{Q}$ which is also evaluation at 1. It is easy to see that there is no $\mathbb{Q}[x]$-module map $\mathbb{Q}[x, x^{-1}] \to \mathbb{Q}[x]$ which makes the diagram commutative, since the image of 1 would have to be divisible by any x^n.

Part II. Prove the following statements (10 points each):

1. Let V and W be irreducible R-modules. Suppose there exist non-zero elements $v \in V$ and $w \in W$ such that (v, w) generates a proper submodule of $V \oplus W$. Then $V \cong W$.

Solution. Let U be the submodule generated by (v, w). Consider the the projections $\pi_V : V \oplus W \to V$ and $\pi_W : V \oplus W \to W$ along the direct sum. Note that $\operatorname{im}(\pi_V|U) = V$ since the image contains v which generates V as V is irreducible. Similarly $\operatorname{im}(\pi_W|U) = W$. Now, if $V \ncong W$, then U has both V and W as composition factors, and so $U = V \oplus W$ by the Jördan-Holder Theorem.

2. Let V be a 7-dimensional vector space over $\mathbb{C}$.

(1) How many similarity classes of linear transformations on V have characteristic polynomial $(x-1)^4(x-2)^3$?

(2) Of the similarity classes in (a), how many have minimal polynomial $(x-1)^2(x-2)^2$?

(1) Partitions of 4: $4, 3+1, 2+2, 2+1+1, 1+1+1+1$, 5 in total. Partitions of 3: $3, 2+1, 1+1+1$, 3 in total. So there are 15 similarity classes of such linear transformations.

(2) Need biggest Jordan blocks of size 2. So just partitions $2+2$ or $2+1+1$ for eigenvalue 1 and $2+1$ for eigenvalue 2, giving just 2 classes in total.

3. Let R be a domain and F be its field of fractions. Prove that $\mathbb{F}$ is an injective R-module.

Solution. Let L be an ideal in R, and $\varphi : L \to \mathbb{F}$ be an R-module homomorphism. We may assume that $L \neq 0$. Pick any $r \in L \setminus \{0\}$ and extend φ to R by sending $s \in R$ to $s\varphi(r)/r \in \mathbb{F}$. To see that the map is well-defined, note that $f(r_1 r_2) = r_1 f(r_2) = r_2 f(r_1)$ implies that $f(r_1)/r_1 = f(r_2)/r_2$. It is easy to see that the map is an R-module map extending φ. It remains to apply Baer's Criterion.

4. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as rings.

Solution. Define a map $\varphi : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}, (\frac{a}{b}, \frac{c}{d}) \mapsto \frac{ab}{bd}$. It is balanced, hence it induces a map $f : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$. It is clear that f is a ring homomorphism. To prove it is an isomorphism, we write down a two-sided inverse map: $g : \frac{a}{b} \mapsto 1 \otimes \frac{a}{b}$. Clearly $f \circ g$ is the identity. To prove that $g \circ f$ is the identity, consider pure tensor $\frac{a}{b} \otimes \frac{c}{d} = b\frac{a}{b} \otimes \frac{c}{bd} = a \otimes \frac{c}{bd} = 1 \otimes \frac{ac}{bd}$. It follows that any element of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ (not just pure tensors) can be written as $1 \otimes \frac{a}{b}$. Now on such an element it is clear that $g \circ f$ is the identity.

5. Working over an algebraically closed field $\mathbb{F}$, prove that the circle $x^2 + y^2 = 1$ and $\mathbb{A}^1$ are isomorphic (as algebraic sets) if and only if $\operatorname{char}\mathbb{F} = 2$.

Solution. If $\operatorname{char}\mathbb{F} = 2$, we have $x^2 + y^2 - 1 = (x + y + 1)^2$, and the result is clear. Let $\operatorname{char}\mathbb{F} \neq 2$. Since $x^2 + y^2 - 1$ is then irreducible, we have that $\mathbb{F}[S^1] = \mathbb{F}[X, Y]/(x^2+y^2-1)$, where $S^1 = \mathcal{V}(x^2 + y^2 - 1)$. So we just need to prove that $\mathbb{F}[x, y]/(x^2 + y^2 - 1)$ is not isomorphic to $\mathbb{F}[T]$.

Let $\varphi : \mathbb{F}[x, y]/(x^2 + y^2 - 1) \to \mathbb{F}[T]$ be an isomorphism. Then the images of x and y are polynomials f and g such that $f^2 + g^2 = 1$. So $(f + ig)(f - ig) = 1$ where $i^2 = -1$. This shows that both polynomials f+ig and f-ig are constant.

Second solution. Assume that $\varphi : \mathbb{F}[x, y]/(x^2+y^2-1) \to \mathbb{F}[T]$ is an isomorphism. Note that every element of $\mathbb{F}[x, y]/(x^2 + y^2 - 1)$ can be written as $\alpha(\bar{x}) + \bar{y}\beta(\bar{x})$ for unique polynomials α, β, where $\bar{x} := x + (x^2 + y^2 - 1)$ and $\bar{y} := y + (x^2 + y^2 - 1)$. This shows in particular that $\bar{x}$ and $\bar{y}$ are irreducible elements of the UFD isomorphic to $\mathbb{F}[T]$. Since φ is an isomorphism, it follows that $\varphi(\bar{x}) = aT + b$ and $\varphi(\bar{y}) = cT + d$ with $a, c \neq 0$. Now

$$\begin{aligned}(cT + d)^2 &= \varphi(\bar{y})^2 = \varphi(\bar{y}^2) = \varphi(1 - \bar{x}^2)\\ &= 1 - (aT + b)^2 = (1 - aT - b)(1 + aT + b).\end{aligned}$$

Now it is easy to see using that $\operatorname{char}\mathbb{F} \neq 2$ that the polynomial in the right hand side has two distinct roots, which leads to a contradiction.

Third solution. Let S be the circle and

$$t \mapsto (f(t), g(t)), \mathbb{A}^1 \to S$$

be an isomorphism. Let a, b, c, d be the points which are mapped to $(0, -1), (0, 1), (1, 0), (-1, 0)$, respectively. Then f has only two roots, namely a and B, and similarly for g, so we can write $f(t) = r(t-a)(t-b)$ for some scalar r and $g(t) = s(t-c)(t-d)$ for some scalar s. Moreover, we also get

$$s(a-c)(a-d) = -1,\ s(b-c)(b-d) = 1,\ r(c-a)(-b) = 1,\ r(d-a)(d-b) = -1.$$

Comparing the first with the third and the second with the fourth equations gives

$$s(a - d) = r(c - b),\ s(c - b) = r(a - d),$$

whence $r^2 = s^2$. So $r^2 = s^2$. On the other hand for $f(t)^2 + g(t)^2 = 1$ only if $r^2 + s^2 = 0$, which is a contradiction unless $\operatorname{char}\mathbb{F} = 2$. If the characteristic is 2, then the two are clearly isomorphic, since the 'circle' becomes a line $x + y = 1$.