ALGEBRA QUALIFYING EXAM, WINTER 2017

Your Name:

Conventions: all rings and algebras are assumed to be unital.

Part I. True or false? If true provide a brief explanation, if false provide a counterexample (10 points each):

1. Let $\mathbf{A}, \mathbf{B}$ be categories and A_1, A_2 be objects of $\mathbf{A}$. If $\mathcal{F}, \mathcal{G} : \mathbf{A} \to \mathbf{B}$ are isomorphic functors and $f, g \in \mathrm{Hom}_{\mathbf{A}}(A_1, A_2)$, then $\mathcal{F}(f) = \mathcal{F}(g)$ if and only if $\mathcal{G}(f) = \mathcal{G}(g)$.
2. If A is a finite dimensional commutative algebra over a field then all irreducible A-modules are 1-dimensional.
3. Let $A \supseteq R$ be an integral ring extension, and assume that A is a domain. If every non-zero prime ideal of R is a maximal ideal, then every non-zero prime ideal of A is also maximal.
4. If β is algebraic over $F(\alpha)$ and β is transcendental over F then α is algebraic over $F(\beta)$.
5. $\mathbb{Q}[x, x^{-1}]$ is a projective $\mathbb{Q}[x]$-module.

Part II. Prove the following statements (10 points each):

1. Let V and W be irreducible R-modules. Suppose there exist non-zero elements $v \in V$ and $w \in W$ such that (v, w) generates a proper submodule of $V \oplus W$. Then $V \cong W$.
2. Let V be a 7-dimensional vector space over $\mathbb{C}$.
   (1) How many similarity classes of linear transformations on V have characteristic polynomial $(x-1)^4(x-2)^3$?
   (2) Of the similarity classes in (a), how many have minimal polynomial $(x-1)^2(x-2)^2$?
3. Let R be a domain and F be its field of fractions. Prove that $\mathbb{F}$ is an injective R-module.
4. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as rings.
5. Working over an algebraically closed field $\mathbb{F}$, prove that the circle $x^2 + y^2 = 1$ and $\mathbb{A}^1$ are isomorphic (as algebraic sets) if and only if $\operatorname{char} \mathbb{F} = 2$.