Fall 2017 Probability Qualifying Exam

September 18, 2017

Instructions:

1. Read all questions carefully. If you are confused ask me!
2. You should have 2 pages including this page. Make sure you have the right number of pages.
3. Unless otherwise noted all conventions and notation follow that of Çinlar. If you are confused about notation, ask!
4. If necessary you may use the back of pages.

Name:______________________________

UO ID:______________________________

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	10	10	10	10	10	10	10	80
Score:									

[10 pts] 1. Suppose $X : (\Omega, \mathcal{H}) \to (E, \mathcal{E})$ is a random variable taking values in the measurable space $(E, \mathcal{E})$. Show that $V \in \sigma X$ iff there exists some measureable function $f : (E, \mathcal{E}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that $V = f \circ X$.

[10 pts] 2. Suppose X_n are real-valued random variables with mean 0 and set $S_n = X_1 + \cdots + X_n$. Show that for every $a > 0$,

$$\mathbb{P}\{\max_{k \leq n} |S_k| > a\} \leq \frac{E[S_n^2]}{a^2}.$$

[10 pts] 3. Suppose (μ_n) is a tight sequence of probability measures on $\mathbb{R}$. Show that every subsequence of (μ_n) has a further subsequence that is weakly convergent. If you use Helly's theorem, give the precise statement, though you need not prove it.

4. Given probability space $(\Omega, \mathcal{H}, \mathbb{P})$, and sub-sigma algebra $\mathcal{F} \subset \mathcal{H}$.

[3 pts] (a) What does it mean for $\mathbb{P}_{\mathcal{F}}$ to have a regular version? Be precise and define all relevant terms.

[7 pts] (b) Suppose $\mathbb{P}_{\mathcal{F}}$ has a regular version Q and X is a real-valued random variable in L^1. Show that a version of $E_{\mathcal{F}}[X]$ is given by

$$\omega \mapsto \int_{\Omega} X(\omega')\, Q_{\omega}(d\omega')$$

5. Let $0 < T_1 < T_2 < \cdots$ be a sequence of random times and define the process $N = (N_t)$ by

$$N_t = \sum_{n=1}^{\infty} 1_{[0,t]} \circ T_n$$

Let $\mathcal{F}$ be the filtration generated by N.

[3 pts] (a) Show that T_k is a stopping time of $\mathcal{F}$.

[7 pts] (b) Suppose $T_k = t_1 + \cdots + t_k$ for some independency of positive random variables $\{t_j\}$. Suppose $E[t_j] = 1/j^2$. Suppose $\mathrm{Var}(t_j) = \frac{1}{j^6}$ and show that with probability 1 there is a random time T such that T is almost surely finite, and $N_T = +\infty$ almost surely.

[10 pts] 6. Let $\mathbb{T} = \mathbb{N}$ and suppose F is a bounded predictable process with respect to filtration $\mathcal{F}$. Show that if M is a martingale with repect to $\mathcal{F}$ then so too is $X = \int F\, dM$.

[10 pts] 7. Let X be a martingale and ζ a stopping time. Show that $\widehat{X} = (X_{t \wedge \zeta})$ is a martingale.

8. Let $W = (W_t)$ be a standard Wiener process.

[5 pts] (a) Show that $W^2 - t$ is a martingale.

[5 pts] (b) Let $T_a = \inf\{t > 0 : |W_t| \geq a\}$. Compute $E[T_a]$.

End of Exam