QUALIFYING EXAM, Fall 2017

Algebraic Topology and Differential Geometry

1. Algebraic Topology

Problem 1.1. State the Theorem which determines the homology groups $\widetilde{H}_q(S^n \setminus S^k)$, where $1 \leq k \leq n-1$. Let $X \subset S^n$ be homeomorphic to $S^p \vee S^q$, $1 \leq p, q \leq n-1$. Compute the homology groups $\widetilde{H}_q(S^n \setminus X)$.

Solution. First, we state the Jordan-Brouwer Theorem:

Theorem 1. *Let $S^{n-1} \subset S^n$ be an embedded sphere in S^n. Then the complement $X = S^n \setminus S^{n-1}$ has two path-connected components: $X = X_1 \sqcup X_2$, where X_1, X_2 are open in S^n. Furthermore, $\partial \overline{X}_1 = \partial \overline{X}_2 = S^{n-1}$.*

Recall the following fact:

Theorem 2. *Let $S^k \subset S^n$, $0 \leq k \leq n-1$. Then*

$$\widetilde{H}_q(S^n \setminus S^k) \cong \begin{cases} \mathbf{Z}, & \text{if } q = n-k-1, \\ 0 & \text{if } q \neq n-k-1. \end{cases} \tag{1}$$

Denote $X_p := S^n \setminus S^p$ and $X_q := S^n \setminus S^q$, where $S^p \vee S^q \subset S^n$. Then

$$\begin{aligned} X_p \cup X_q &= S^n \setminus (S^p \cap S^q) = S^n \setminus * \cong \mathbf{R}^n, \\ X_p \cap X_q &= S^n \setminus (S^p \cup S^q) = S^n \setminus (S^p \vee S^q) = X. \end{aligned}$$

Then the Mayer-Vietoris exact sequence gives

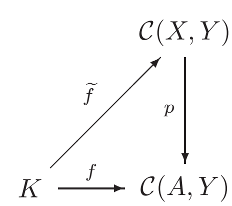
$$\tilde{H}_j(X) \cong \tilde{H}_j(X_p) \oplus \tilde{H}_j(X_q) = \begin{cases} \mathbf{Z}, & \text{if } p \neq q \text{ and } j = n-p-1 \text{ or } j = n-q-1, \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if } p = q \text{ and } j = n-p-1, \\ 0 & \text{if } q \neq n-k-1. \end{cases}$$

Problem 1.2. Let (X, A) be a CW-pair, and Y is a CW-complex as well. Let $E = \mathcal{C}(X, Y)$, $B = \mathcal{C}(A, Y)$, and the map $p : E \to B$ be defined as

$$p : (f : X \to Y) \mapsto (f|_A : A \to Y).$$

Prove that the map $p : E \to B$ is a Serre fiber bundle.

Solution. We have to show that the map $p : \mathcal{C}(X, Y) \to \mathcal{C}(A, Y)$ satisfies the Lifting Homotopy property. Let K be a CW-complex, and $f : K \to \mathcal{C}(A, Y)$, $\tilde{f} : K \to \mathcal{C}(X, Y)$ be such that the diagram



commutes. Let $F : K \times I \to \mathcal{C}(A, Y)$ be a homotopy such that $F|_{K \times \{0\}} = f$. In particular,

$$F \in \mathcal{C}(K \times I, \mathcal{C}(A, Y))$$

We notice that the map $p : \mathcal{C}(X,Y) \to \mathcal{C}(A,Y)$ induces the map of the space of continuous map $p_\# : \mathcal{C}(K \times I, \mathcal{C}(X,Y)) \to \mathcal{C}(K \times I, \mathcal{C}(A,Y))$. Furthemore, there are natural homepmorphisms:

$$\mathcal{C}(K \times I, \mathcal{C}(X,Y)) \cong \mathcal{C}(K \times X \times I, Y), \quad \mathcal{C}(K \times I, \mathcal{C}(A,Y)) \cong \mathcal{C}(K \times A \times I, Y).$$

We obtain a map $p_\# : \mathcal{C}(K \times X \times I, Y) \to \mathcal{C}(K \times A \times I, Y)$. The homotopy $F : K \times I \to \mathcal{C}(A,Y)$ gives the homotopy $F' : K \times A \times I \to Y$ by the formula

$$F' : (k,a,t) \to F(k,t)(a) \in Y.$$

Here $F'|_{K \times A \times \{0\}} = f'$, where $f' : (k,a) \mapsto f(k)(a)$. Let $\tilde{f}' : K \times X \to Y$ be the map given as $\tilde{f}' : (k,x) \mapsto \tilde{f}(k)(x)$. Since the pair $(K \times X, K \times A)$ is CW-pair, it is a Borsuk pair. Thus any map $F' : K \times A \times I \to Y$ such that $F'|_{K \times A \times \{0\}} = f'$, where $f' = \tilde{f}'|_A$ extends to a map $\tilde{F}' : K \times X \times I \to Y$ such that $\tilde{F}'|_{K \times A \times I} = F'$ and $\tilde{F}'|_{K \times X \times \{0\}} = \tilde{f}'$.

Problem 1.3. Define the Hopf invariant. Prove that the group $\pi_{4n-1}(S^{2n})$ has infinite order.

Solution. Let $\alpha \in \pi_{4n-1}(S^{2n})$, and let $f : S^{4n-1} \to S^{2n}$ be a representative of α. Let $X_\alpha = S^{2n} \cup_f D^{4n}$. It is easy to compute the cohomology groups of X_α:

$$H^q(X_\alpha; \mathbf{Z}) = \begin{cases} \mathbf{Z}, & q = 0, 2n, 4n, \\ 0, & \text{otherwise.} \end{cases}$$

Let $a \in H^{2n}(X_\alpha; \mathbf{Z})$, $b \in H^{4n}(X_\alpha; \mathbf{Z})$ be generators. Since $a^2 = a \cup a \in H^{4n}(X_\alpha; \mathbf{Z})$, then $a^2 = h \cdot b$, where $h \in \mathbf{Z}$. The number $h(\alpha) = h$ is the **Hopf invariant** of the element $\alpha \in \pi_{4n-1}(S^{2n})$. We use the following fact:

Lemma 3. *(1) $h(\alpha) = 0$ if $\alpha = 0$;*
(2) $h(\varphi_1) + h(\varphi_2) = h(\varphi_1 + \varphi_2)$.

We obtain a homomorphism $h : \pi_{4n-1}(S^{2n}) \to \mathbf{Z}$. Let $\iota_{2n} \in \pi_{2n}(S^{2n})$ be a generator given by the identity map $S^{2n} \to S^{2n}$. Then $[\iota_{2n}, \iota_{2n}]$ denote the Whitehead product.

Lemma 4. *The Hopf invariant is not trivial, in particular,*

$$h([\iota_{2n}, \iota_{2n}]) = 2.$$

First we compute the cohomology (together with a product structure) $H^q(S^{2n} \times S^{2n})$:

$$H^q(S^{2n} \times S^{2n}; \mathbf{Z}) = \begin{cases} \mathbf{Z}, & q = 0, 4n, \\ \mathbf{Z} \oplus \mathbf{Z}, & q = 2n \\ 0, & \text{otherwise.} \end{cases}$$

Let $c_1, c_2 \in H^{2n}(S^{2n} \times S^{2n})$ be such generators that the homomorphisms

$$H^{2n}(S_1^{2n}) \xrightarrow{p_1^*} H^{2n}(S_1^{2n} \times S_2^{2n}) \xleftarrow{p_2^*} H^{2n}(S_2^{2n})$$

induced by the projections $S_1^{2n} \xleftarrow{p_1} S_1^{2n} \times S_2^{2n} \xrightarrow{p_2} S_2^{2n}$, send the generators c_1 and c_2 to the generators of the groups $H^{2n}(S_1^{2n})$, $H^{2n}(S_2^{2n})$. Let $d \in H^{4n}(S^{2n} \times S^{2n})$ be a generator. Then $c_1 c_2 = d$. We also note that $c_1^2 = 0$ and $c_2^2 = 0$ since by naturality $p_1^*(c_1)^2 = 0$ and $p_2^*(c_2)^2 = 0$. So we have that the ring $H^*(S_1^{2n} \times S_2^{2n})$ is generated over $\mathbf{Z}$ by the elements 1, c_1, c_2 with the relations $c_1^2 = 0$, $c_2^2 = 0$. In particular, we have:

$$(c_1 + c_2)^2 = c_1^2 + 2c_1c_2 + c_2^2 = 2d.$$

Proof of Lemma 4. We consider the factor space

$$X = S^{2n} \times S^{2n} / \sim,$$

where we identify the points $(x, x_0) = (x_0, x)$, where x_0 is the base point of S^{2n}. We notice that

The space $X = S^{2n} \times S^{2n}/\sim$ *is homeomorphic to the space* $S^{2n} \cup_f D^{4n}$, *where* f *is the map defining the Whitehead product* $[\iota_{2n}, \iota_{2n}]$.

Indeed $S^{2n} \times S^{2n} = (S^{2n} \vee S^{2n}) \cup_w D^{4n}$, where w is the map we described above. The generator ι_{2n} is represented by the identical map $S^{2n} \to S^{2n}$. The composition

$$S^{4n-1} \xrightarrow{w} S^{2n} \vee S^{2n} \xrightarrow{Id \vee Id} S^{2n}$$

represents the element $[\iota_{2n}, \iota_{2n}]$. It exactly means that the identification $(S^{2n}, x_0) = (x_0, S^{2n})$ we just did in the space $S^{2n} \times S^{2n}$ is the same as to attach D^{4n} with the attaching map $(Id \vee Id) \circ w$.

Problem 1.4. Let $n \geq 1$. Consider the map

$$g : S^{4n-2} \times S^5 \xrightarrow{\text{proj}} (S^{4n-2} \times S^5)/(S^{4n-2} \vee S^5) = S^{4n+3} \xrightarrow{\text{Hopf}} \mathbf{HP}^n.$$

Prove that g induces trivial homomorphism in homology and homotopy groups, however g is not homotopic to a constant map.

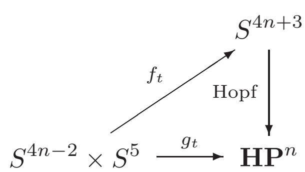
Solution. Consider the homomorphism $g_* : \pi_q(S^{4n-2} \times S^5) \xrightarrow{\text{proj}_*} \pi_q S^{4n+3} \xrightarrow{\text{Hopf}_*} \pi_q \mathbf{HP}^n$. Since $\pi_q(S^{4n-2} \times S^5) \cong \pi_q S^{4n-2} \oplus \pi_q S^5$, the homomorphism proj_* factors through the projections

$$\pi_q S^{4n+3} \leftarrow \pi_q S^{4n-2} \xleftarrow{\text{proj}_1} \pi_q(S^{4n-2} \times S^5) \xrightarrow{\text{proj}_2} \pi_q S^5 \to \pi_q S^{4n+3}$$

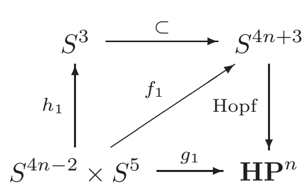
Since the maps $S^{4n-2} \to S^{4n+3}$ and $S^5 \to S^{4n+3}$ are homotopically trivial, the homomorphism $g_* : \pi_q(S^{4n-2} \times S^5) \to \pi_q S^{4n+3} \to \pi_q \mathbf{HP}^n$ is trivial as well.

The homomorphism $g_* : \tilde{H}_q(S^{4n-2} \times S^5) \xrightarrow{\text{proj}_*} \tilde{H}_q S^{4n+3} \xrightarrow{\text{Hopf}_*} \tilde{H}_q \mathbf{HP}^n$ is also trivial. Indeed, the group $\tilde{H}_{4n+3} S^{4n+3} = \mathbf{Z}$ and trivial ortherwise and $\tilde{H}_q \mathbf{HP}^n$ is non-trivial only if $q = 4, \ldots, 4n$, i.e., any homomorphism $\tilde{H}_q S^{4n+3} \to \tilde{H}_q \mathbf{HP}^n$ is trivial.

Now we assume that the map $g : S^{4n-2} \times S^5 \to \mathbf{HP}^n$ is contractible. Let $g_t : S^{4n-2} \times S^5 \to \mathbf{HP}^n$ be a homotopy such that $g_0 = g$ and $g_1(S^{4n-2} \times S^5) = x_0 \in \mathbf{HP}^n$. By the Lifting Homotopy Property, there exists a homotopy $f_t : S^{4n-2} \times S^5 \to S^{4n+3}$ such that the diagram



commutes. Here $f_0 = \text{proj}$ as above. Then we have that $f_1(S^{4n-2} \times S^5) \subset \text{Hopf}^{-1}(x_0) = S^3$. We notice that the homomorphism $\text{proj}_* : H_{4n+3}(S^{4n-2} \times S^5) \to H_{4n+3}(S^{4n+3})$ is non-trivial (it is an isomorphism). However $(f_1)_*$ must be trivial since the map f_1 factors through S^3:



Since $(f_1)_* = \text{proj}_*$, we obtain a contradiction.

Problem 1.5. (i) Let $f : \mathbf{RP}^{4n} \times \mathbf{CP}^{2n} \to \mathbf{RP}^{4n} \times \mathbf{CP}^{2n}$ be a continuous map. Prove that f always has a fixed point.

(ii) Give an example of a map $g : \mathbf{RP}^{2k} \times \mathbf{CP}^k \to \mathbf{RP}^{2k} \times \mathbf{CP}^k$ without fixed points.

Solution. (i) We compute the cohomology groups $H^*(\mathbf{RP}^{4n}\times\mathbf{CP}^{2n};\mathbf{Q})$:

$$H^j(\mathbf{RP}^{4n}\times\mathbf{CP}^{2n};\mathbf{Q})=\begin{cases}\mathbf{Q}, & q=0,2,\cdots,4n,\\ 0, & \text{otherwise.}\end{cases}$$

Indeed, we have that $\tilde{H}^j(\mathbf{RP}^{4n};\mathbf{Q})=0$. Then the Künneth formula implies the result. Furthermore, we see that the projection $\mathbf{RP}^{4n}\times\mathbf{CP}^{2n}\to\mathbf{CP}^{2n}$ induces an isomorphism

$$H^*(\mathbf{CP}^{2n};\mathbf{Q})\cong H^*(\mathbf{RP}^{4n}\times\mathbf{CP}^{2n};\mathbf{Q})\cong\mathbf{Q}[z]/z^{2n+1},$$

where $z\in H^2(\mathbf{RP}^{4n}\times\mathbf{CP}^{2n};\mathbf{Q})$. Let $f:\mathbf{RP}^{4n}\times\mathbf{CP}^{2n}\to\mathbf{RP}^{2k}\times\mathbf{CP}^{2n}$ be a map. Then $f^*(z)=\lambda\cdot z\in H^2(\mathbf{RP}^{2k}\times\mathbf{CP}^{2n};\mathbf{Q})$. Then we have that $f^*(z^k)=\lambda^k\cdot z^k$. We obtain the Lefcshetz number $L(f)=1+\lambda+\cdots+\lambda^{2n}$. If $\lambda=1$, then $L(f)\neq 0$. If $\lambda\neq 1$, we have $L(f)=\frac{\lambda^{2n+1}-1}{\lambda-1}\neq 0$. Since $\mathbf{RP}^{2k}\times\mathbf{CP}^{2n}$ is a finite CW-complex, Lefcshetz fixed-point theorem implies that f has a fixed point.

(ii) Consider $\mathbf{CP}^1=S^2$ and the intipodal map $A:x\mapsto -x$: this map does not have fixed points. By construction, the map

$$Id\times A:\mathbf{RP}^2\times\mathbf{CP}^1\to\mathbf{RP}^2\times\mathbf{CP}^1$$

does not have fixed points.

Problem 1.6. Compute the homotopy group $\pi_q(S^2\vee S^2)$ for $q=1,2,3$.

Solution. We have that $\pi_1(S^2\vee S^2)=0$ since $S^2\vee S^2$ has a CW-decomposition with one zero cell and two 2-cells. Then the CAT iplies that any map $S^1\to S^2\vee S^2$ is homotopic to a constant map. Then we know that $\pi_2(S^2\times S^2)\cong\pi_2(S^2)\oplus\pi_2(S^2)=\mathbf{Z}\oplus\mathbf{Z}$. We notice that the inclusion map $i:S^2\vee S^2\to S^2\times S^2$ induces an isomorphism $\pi_2(S^2\vee S^2)\to\pi_2(S^2\times S^2)$. Indeed, we have that

$$S^2\times S^2=(S^2\vee S^2)\cup_w D^4,$$

where $w:S^3\to S^2\vee S^2$ is the Whitehead map.

Now we have that the homomorphism $i_*:\pi_3(S^2\vee S^2)\to\pi_3(S^2\times S^2)$ has the kernel given by the attaching map $w:S^3\to S^2\vee S^2$. Thus we have the exact sequence

$$\mathbf{Z}\to\pi_3(S^2\vee S^2)\to\pi_3(S^2\times S^2)$$

Thus we conclude $\pi_3(S^2\vee S^2)\cong\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}$.

Problem 1.7. Let M^2_g be a two-dimensional surface of genus $g\geq 1$ (oriented). Compute the homotopy groups $\pi_q(M^2_g)$ for all $q\geq 1$. Hint: use covering spaces.

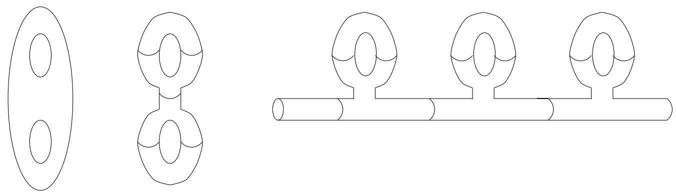
Solution. The standard model for M^2_g is $4g$-gone labeled as $a_j,b_j,\bar{a}_j,\bar{b}_j$, $j=1,\dots,g$. Here the notations $\bar{a}_j,\bar{b}_j$ mean that a_j, b_j are identified with the sides $\bar{a}_j,\bar{b}_j$ in the opposite direction. Let $F(a_j,b_j,\bar{a}_j,\bar{b}_j|j=1,\dots,g)$ denote a free group with generators $a_j,b_j,\bar{a}_j,\bar{b}_j|j=1,\dots,g$. Thus the fundmental group $\pi_1(M^2_g)$ is isomorphic to the group

$$F(a_j,b_j|j=1,\dots,g)/a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}.$$

Now we prove the following

Theorem 5. *$\pi_q(M^2_g)=0$ for all $q\geq 2$.*

Proof. Let $g=1$ then $M^2_g=T^2$, and we have the univesral covering space $p:\mathbf{R}^2\to T^2$, where $p:(\theta_1,\theta_2)\mapsto(e^{i\pi\theta_1},e^{i\pi\theta_2})\in T^2$. Then $p_*:\pi_q(\mathbf{R}^2)\to\pi_q(T^2)$ induces isomorphism for all $q\geq 2$. Now, by induction, assume that $\pi_q(M^2_g)=0$ for all $q\geq 2$ for some $g\geq 1$. Then we consider the following covering $p:Z_g\to M^2_{g+1}$. Here Z_g is given by an infinite cylinder $\mathbf{R}\times S^1$, where every coordinate $(k/2,*)\in\mathbf{R}\times S^1$ is used to take a connected sum with M^2_g, see the figure below, where $g=1$:



Then we have a map $p : Z_g \to M^2_{g+1}$ which sends $(x, \alpha) \mapsto (e^{i\pi x}, \alpha)$. It is easy to see that p is a covering. Let $f : S^q \to Z_g$ be a map. The image of f is compact, so $f(S^q) \subset Y_g \subset Z_g$, where Y_g has only $[-N, N] \times S^1$ with connected sums of M^2_g as before. Then Y_g is nothi g but an oriented surface $M^2_{\bar{g}}$ for some $\bar{g}$ with two disks removed. Such space Y_g is homotopy equivalent to one-point union of circles. Thus the map $f : S^q \to Y_q \to Z_q$ is homotopically trivial, i.e., this shows that $\pi_q M^2_{g+1} = 0$.

Problem 1.8. Let $T^2 = S^1 \times S^1$ be a torus. Prove that any map $f : \mathbf{RP}^2 \to T^2$ is homotopic to a constant map.

Solution. We consider the universal covering $p : \mathbf{R}^2 \to T^2$. Let $f : \mathbf{RP}^2 \to T^2$ be a map, where a base point $x_0 \in \mathbf{RP}^2$ maps to $y_0 = f(x_0)$. Let $\tilde{y}_0 \in p^{-1}(y_0)$ be a point in $\mathbf{R}^2$. We know that $\pi_1 \mathbf{RP}^2 = \mathbf{Z}_2$, and $\pi_1 T_2 = \mathbf{Z} \oplus \mathbf{Z}$. Thus $f_*(\pi_1 \mathbf{RP}^2) = 0 \in \pi_1 T_2$ for any map $f : \mathbf{RP}^2 \to T^2$. We use the following result:

Theorem 6. *Let $p : T \to X$ be a covering space, and Z be a path-connected space, $x_0 \in X$, $\tilde{x}_0 \in T$, $p(\tilde{x}_0) = x_0$. Given a map $f : (Z, z_0) \to (X, x_0)$ there exists a lifting $\tilde{f} : (Z, z_0) \to (T, \tilde{x}_0)$ if and only if $f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(T, \tilde{x}_0))$.*

In our case, it means that there exists a map $\tilde{f} : \mathbf{RP}^2 \to \mathbf{R}^2$ such that $\tilde{f}(x_0) = \tilde{y}_0$ and $\tilde{f} = p \circ f$. Thus the homotopy class of f is in the image of $p_* : \pi_1 \mathbf{R}^2 \to T^2$. Hence $f \sim 0$.

Problem 1.9. Find $H_1(X; \mathbf{Z})$ and $H^1(X; \mathbf{Z})$ in terms of the group π, where π is a finite group.

Solution. We have that $H_1(X; \mathbf{Z}) = \pi/[\pi, \pi]$. If $\pi/$ is finite group, then $\pi/[\pi, \pi]$ is finite abelian group. The universal coefficient theorem yields

$$H^1(X; \mathbf{Z}) \cong \mathrm{Ext}(H_0(X, \mathbf{Z}), \mathbf{Z}) \oplus \mathrm{Hom}(H_1(X, \mathbf{Z}), \mathbf{Z})$$

Then $\mathrm{Ext}(H_0(X, \mathbf{Z}), \mathbf{Z}) = 0$ since $H_0(X, \mathbf{Z})$ is free abelian, and $\mathrm{Hom}(H_1(X, \mathbf{Z}), \mathbf{Z}) = 0$ since $H_1(X, \mathbf{Z})$ is finite. Thus $H^1(X; \mathbf{Z}) = 0$.

Problem 1.10. Prove that the spaces $\mathbf{CP}^\infty \times S^3$ and S^2 have isomorphic homotopy groups and that they are not homotopy equivalent.

Solution. First, we claim that $\pi_2 \mathbf{CP}^\infty = \mathbf{Z}$, and $\pi_q \mathbf{CP}^\infty = 0$ for all $q \neq 0$. Indeed, consider the Hopf fibre bunde $S^{2n+1} \to \mathbf{CP}^n$ with the fibre S^1. Then the exact sequence in homotopy groups

$$\cdots \to \pi_q(S^1) \to \pi_q(S^{2n+1}) \to \pi_q(\mathbf{CP}^n) \to \pi_{q-1}(S^1) \to \cdots$$

imlies that $\pi_2 \mathbf{CP}^\infty = \mathbf{Z}$ and $\pi_q \mathbf{CP}^\infty = 0$ for all $q \neq 0$ and $q < 2n + 1$. Now we hve that

$$\pi_q(\mathbf{CP}^\infty \times S^3) \cong \pi_q \mathbf{CP}^\infty \oplus \pi_q S^3 = \begin{cases} \mathbf{Z}, & q = 2, \\ \pi_q S^3, & q \neq 2 \end{cases}$$

The Hopf bundle $S^3 \to S^2$ gives us that $\pi_q S^2 \cong \pi_q S^3$ for $q \geq 3$. We conclude that the homotopy groups of the spaces $\mathbf{CP}^\infty \times S^3$ and S^2 have isomorphic homotopy groups. Now we assume that there is a homotopy equivalence $f : S^2 \to \mathbf{CP}^\infty \times S^3$. Then f has to induce an isomorphism in all homotopy groups. Then $f = (f_1, f_2)$, where $f_1 : S^2 \to \mathbf{CP}^\infty$, and $f_2 : S^2 \to S^3$. However, f_2 is homotopically trivial, and for $q \geq 3$, we have that a map $\varphi : S^q \to S^2$ maps to the composition

$$S^q \xrightarrow{\varphi} S^2 \xrightarrow{f} S^3$$

which is homotopically trivial.