1	2	3	4	5	6	7	8	9	10	
11	12	13	14	15	16	17	18	19	20	Total

QUALIFYING EXAM, Fall 2017

Algebraic Topology and Differential Geometry

NAME ____

(PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ____________________ **SIGNATURE** ____________________

Please do any 10 problems out of the following 20.

1. Algebraic Topology

Problem 1.1. State the Theorem which determines the homology groups $\widetilde{H}_j(S^n \setminus S^k)$, where $1 \leq k \leq n-1$. Let $X \subset S^n$ be homeomorphic to $S^p \vee S^q$, $1 \leq p, q \leq n-1$. Compute the homology groups $\widetilde{H}_j(S^n \setminus X)$.

Problem 1.2. Let (X, A) be a CW-pair, and Y is a CW-complex as well. Let $E = \mathcal{C}(X, Y)$, $B = \mathcal{C}(A, Y)$, and the map $p : E \to B$ be defined as

$$p : (f : X \to Y) \mapsto (f|_A : A \to Y).$$

Prove that the map $p : E \to B$ is a Serre fiber bundle.

Problem 1.3. Define the Hopf invariant. Prove that the group $\pi_{4n-1}(S^{2n})$ has infinite order.

Problem 1.4. Let $n \geq 1$. Consider the map

$$g : S^{4n-2} \times S^5 \xrightarrow{\text{proj}} (S^{4n-2} \times S^5)/(S^{4n-2} \vee S^5) = S^{4n+3} \xrightarrow{\text{Hopf}} \mathbf{HP}^n.$$

Prove that g induces trivial homomorphism in homology and homotopy groups, however g is not homotopic to a constant map.

Problem 1.5.
(i) Let $f : \mathbf{RP}^{4n} \times \mathbf{CP}^{2n} \to \mathbf{RP}^{4n} \times \mathbf{CP}^{2n}$ be a continuous map. Prove that f always has a fixed point.
(ii) Give an example of a map $g : \mathbf{RP}^{2k} \times \mathbf{CP}^{k} \to \mathbf{RP}^{2k} \times \mathbf{CP}^{k}$ without fixed points.

Problem 1.6. Compute the homotopy group $\pi_q(S^2 \vee S^2)$ for $q = 1, 2, 3$.

Problem 1.7. Let M^2_g be a two-dimensional surface of genus $g \geq 1$ (oriented). Compute the homotopy groups $\pi_q(M^2_g)$ for all $q \geq 1$. Hint: use covering spaces.

Problem 1.8. Let $T^2 = S^1 \times S^1$ be a torus. Prove that any map $f : \mathbf{RP}^2 \to T^2$ is homotopic to a constant map.

Problem 1.9. Find $H_1(X; \mathbf{Z})$ and $H^1(X; \mathbf{Z})$ in terms of the group $\pi = \pi_1(X, x_0)$, where π is a finite group.

Problem 1.10. Prove that the spaces $\mathbf{CP}^\infty \times S^3$ and S^2 have isomorphic homotopy groups and that they are not homotopy equivalent.

2. Differential Geometry

Problem 2.1. Let $T(\varphi, \theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ be "spherical coordinates" on $S^2 = \{ (x, y, z)\ x^2 + y^2 + z^2 = 1 \}$. Let ∇ be the Levi-Civita connection.
(1) For what values of θ are the curves $\sigma_\theta(\varphi) := T(\varphi, \theta)$ geodesics?
(2) For what values of φ are the curves $\tau_\varphi(\theta) := T(\varphi, \theta)$ geodesics?

Problem 2.2. Let $S := \{ (x,y,z) \mid x^4 + 2x^2 + y^4 + 2y^2 + z^4 + 2z^2 = 1 \} \subset \mathbf{R}^3$. Show that S is a smooth surface and that there are at least 9 distinct closed geodesics on S.

Problem 2.3. Let $ds^2 = \frac{dx^2+dy^2}{(1+x^2+y^2)^2}$ be the metric on the whole plane. Determine the Gaussian curvature K.

Problem 2.4. Let κ be the sectional curvature and let ρ be the Ricci curvature of a Riemannian manifold M of dimension $m \geq 2$. Show that there exists a constant $c(m)$ which only depends on the underlying dimension m so that if η is any unit tangent vector in the tangent space to M at a point P of M, then

$$\rho(\eta,\eta) = c(m) \int_{\xi \in S(\eta^\perp)} \kappa(\mathrm{Span}\,\{\eta,\xi\}) d\xi$$

Thus in a certain sense $\rho(\eta,\eta)$ is the normalized average of the sectional curvatures of the 2-panes containing η. Here $S(\eta^\perp)$ is the unit $(m-2)$-dimensional sphere of unit tangent vectors which are perpendicular to η, and $d\xi$ is the usual measure on this sphere.

Problem 2.5. Let $\mathbf{CP}^n$ denote the space of complex lines through the origin in $\mathbf{C}^{n+1}$. This has a canonical structure as a holomorphic manifold. Carefully indicate at each step in the argument the facts that you are using.

(1) Show that $\mathbf{CP}^2$ does not admit an orientation reversing diffeomorphism.
(2) Show that $\mathbf{CP}^3$ is not diffeomorphic to $S^2 \times S^4$.

Problem 2.6. Let M be a compact oriented manifold without boundary with $\dim M = m$. Let $[\omega_p]$ belong to $H^p_{\mathrm{deR}}(M)$ and let $[\sigma_{m-p}]$ belong to $H^{m-p}_{\mathrm{deR}}(M)$. Show the map

$$\mathcal{I} : \omega_p \otimes \sigma_{m-p} \mapsto \int_M \omega_p \wedge \sigma_{m-p}$$

extends to a well defined map $\mathcal{I} : H^p_{\mathrm{deR}}(M) \otimes H^{m-p}_{\mathrm{deR}}(M) \to \mathbf{R}$ in the de Rham cohomology.

Problem 2.7. Prove or disprove the assertion "The $ax+b$ group is unimodular" (i.e. it has bi-invariant differential form of maximal degree). One may identify the $ax+b$ group with the matrix group $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ for $a > 0$ and $b \in \mathbf{R}$.

Problem 2.8. Let $T(x,y) = (-y,x)$ define a fixed point free action of $\mathbf{Z}_4$ on the Cartisian product $S^a \times S^a$. Let $M = (S^a \times S^a)/\mathbf{Z}_4$ be the quotient manifold. Determine the de Rham cohomology of M, and also the ring structure.

Problem 2.9. Suppose that M^2 is compact, and is covered by open sets U_1 and U_2. Suppose also each U_i are regular coordinate balls, that is, each is precompact in an open set $\tilde{U}_i$, and that there exist maps

$$\varphi_i : \tilde{U}_i \to B_1(0) \subset \mathbb{R}^2, \quad i = 1,2$$

that are diffeomorphisms providing charts for M. Using φ_1, φ_2, and possibly some other functions, construct an embedding of M into $\mathbb{R}^6$. Prove that this map is indeed an embedding.

Problem 2.10. Suppose the following. M, N, and P are all smooth manifolds.

$$F : M \to N \text{ is smooth, surjective}$$
$$G : N \to P \text{ is continuous but not } C^1$$

but the map $G \circ F : M \to P$ is smooth.

(1) Give an example of such a situation
(2) Prove that the set $E = \{x \in N : G \text{ is not smooth}\}$ has measure zero.