Fall 2017 Probability Qualifying Exam

September 18, 2017

Instructions:

1. Read all questions carefully. If you are confused ask me!
2. You should have 6 pages including this page. Make sure you have the right number of pages.
3. Unless otherwise noted all conventions and notation follow that of Çinlar. If you are confused about notation, ask!
4. If necessary you may use the back of pages.

Name:________________________________

UO ID:________________________________

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	10	10	10	10	10	10	10	80
Score:									

[10 pts] 1. Suppose $X : (\Omega, \mathcal{H}) \to (E, \mathcal{E})$ is a random variable taking values in the measurable space $(E, \mathcal{E})$. Show that $V \in \sigma X$ iff there exists some measureable function $f : (E, \mathcal{E}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that $V = f \circ X$.

Solution: If $V = f \circ X$ then $V \in \sigma X$.

σX is generated by the p-system $\mathcal{P} = \{X^{-1}(B) : B \subseteq \mathbb{R} \text{ Borel}\}$. Let $H = X^{-1}(B)$ then $\mathbf{1}_H = \mathbf{1}_{X^{-1}(B)} = \mathbf{1}_B \circ X$.

That is, if $\mathcal{C} = \{V \in \sigma X : V = f \circ X \text{ for some measurable } f : E \to \mathbb{R}\ \}$ then $\mathbf{1}_H \in \mathcal{C}$. If V and W are in $\mathcal{C}$ then there exist measurable f and g such that $V = f \circ X$ and $W = g \circ X$. Thus $cV + W = cf \circ X + g \circ X = (cf + g) \circ X \in \mathcal{C}$.

Finally, suppose $V_n \nearrow V$ for $V_n \in \mathcal{C}$. Then there exist measurable functions f_n so that $V_n = f_n \circ X$. We may assume that f_n is identically 0 on the complement of $X(\Omega)$ by multiplying by the appropriate indicator function if necessary. Define f on the image of X by

$$f(x) = \lim f_n(x) \qquad x = X(\omega)$$

If we take f to be identically 0 on the complement of $X(\Omega)$ then $f_n \nearrow f$ and hence f is measurable. It follows that $V = f \circ X$ and thus $\mathcal{C}$ is a monotone class. Since $\mathcal{C}$ contains $\mathcal{P}$ it contains all measurable functions in σX.

[10 pts] 2. Suppose X_n are real-valued random variables with mean 0 and set $S_n = X_1 + \cdots + X_n$. Show that for every $a > 0$,

$$\mathbb{P}\{\max_{k \le n} |S_k| > a\} \le \frac{E[S_n^2]}{a^2}.$$

Solution: This is simply Kolmogorov's inequality. Define the random variable $N(\omega) = \inf\{k : |S_k(\omega)| > a\}$. Note that $\mathbf{1}_{N=k}$ can be given in terms of the random variables $X_1, \ldots, X_k$ and hence $U := \mathbf{1}_{N=k} S_k$ depends only on the variables $X_1, \ldots, X_k$ and is independent of $V := S_n - S_k$. Since $E[V] = 0$, we have

$$0 = E[V]E[U] = E[VU] = E\left[S_k(S_n - S_k)\mathbf{1}_{N=k}\right]$$

Also, $S_n^2 = [S_k + (S_n - S_k)]^2 \ge S_k^2 + 2(S_n - S_k)S_k$ and on the event $\{N = k\}$, $S_k^2 > a^2$. Thus,

$$E[S_n^2 \mathbf{1}_{N=k}] \ge a^2 E[\mathbf{1}_{N=k}] + 2E\left[S_k(S_n - S_k)\mathbf{1}_{N=k}\right] = a^2 \mathbb{P}\{N = k\},$$

and

$$a^2 \mathbb{P}\{N \le n\} \le \sum_{k=1}^{n} E[S_n^2 \mathbf{1}_{N=k}] = E[S_n^2 \mathbf{1}_{N \le n}] \le E[S_n^2] = \mathrm{Var}(S_n).$$

This completes the proof since $\{N \le n\} = \{\max_{k \le n} |S_k| > a\}$.

[10 pts] 3. Suppose (μ_n) is a tight sequence of probability measures on $\mathbb{R}$. Show that every subsequence of (μ_n) has a further subsequence that is weakly convergent. If you use Helly's theorem, give the precise statement, though you need not prove it.

Solution: Let (c_n) be the corresponding sequence of distribution functions of (μ_n). By Helly's Theorem there is a subsequence $(c_{n'})$ and a distribution function c so that $c_{n'}$ converges to c at every point of continuity of c. Without tightness there is no guarentee that c is a distribution function in the sense of probability. That is, it remains to show

$$\lim_{x\to-\infty} c(x) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} c(x) = 1$$

By tightness, there exists a closed interval $[a, b]$ such that $\mu_n[a, b] > 1 - \epsilon$ for all n. Thus, $\mu_n(-\infty, a) < \epsilon$ and $\mu_n(-\infty, b] > 1 - \epsilon$. This implies that $c(a) < \epsilon$ and $c(b) > 1 - \epsilon$. Since ϵ was arbitrary this shows that c is a distribution function in the sense of probability. If μ is the corresponding probability measure on $\mathbb{R}$ then $\mu_{n'}$ converges weakly to μ.

4. Given probability space $(\Omega, \mathcal{H}, \mathbb{P})$, and sub-sigma algebra $\mathcal{F} \subset \mathcal{H}$.

[3 pts] (a) What does it mean for $\mathbb{P}_{\mathcal{F}}$ to have a regular version? Be precise and define all relevant terms.

Solution: $Q(\omega, B)$ is a regular version for $\mathbb{P}_{\mathcal{F}}$ if Q is a probability kernel from $(\Omega, \mathcal{H})$ into $(\Omega, \mathcal{F})$ so that

$$E_{\mathcal{F}}[\mathbf{1}_H](\omega) = Q(\omega, H).$$

[7 pts] (b) Suppose $\mathbb{P}_{\mathcal{F}}$ has a regular version Q and X is a real-valued random variable in L^1. Show that a version of $E_{\mathcal{F}}[X]$ is given by

$$\omega \mapsto \int_\Omega X(\omega')\, Q_\omega(d\omega')$$

Solution: Note that by definition,

$$E_{\mathcal{F}}[\mathbf{1}_H](\omega) = Q(\omega, H) = \int_\Omega \mathbf{1}_H(\omega')\, Q_\omega(d\omega')$$

Monotone class, or definition of conditional expectation with monotone convergence does the rest.

5. Let $0 < T_1 < T_2 < \cdots$ be a sequence of random times and define the process $N = (N_t)$ by

$$N_t = \sum_{n=1}^{\infty} 1_{[0,t]} \circ T_n$$

Let $\mathcal{F}$ be the filtration generated by N.

[3 pts] (a) Show that T_k is a stopping time of $\mathcal{F}$.

Solution: $\{T_k \leq t\} = \{N_t \geq k\} \in \mathcal{F}_t$

[7 pts] (b) Suppose $T_k = t_1 + \cdots + t_k$ for some independency of positive random variables $\{t_j\}$. Suppose $E[t_j] = 1/j^2$. Suppose $\mathrm{Var}(t_j) = \frac{1}{j^6}$ and show that with probability 1 there is a random time T such that T is almost surely finite, and $N_T = +\infty$ almost surely.

Solution: Note that if $t_k < 2/k^2$ for all but finitely many then $T = \sum t_k$ is finite, and $N_T = \infty$. Thus it suffices to show that $t_k \geq 2/k^2$ only finitely often almost surely. By Chebyshev's inequality, for $\sigma_k^2 = \mathrm{Var}(t_k) = 1/k^6$

$$\mathbb{P}\{|t_k - 1/k^2| \geq c\sigma_k\} \leq \frac{1}{c^2}$$

With $c = k$ this becomes

$$\mathbb{P}\{|t_k - 1/k^2| \geq \frac{1}{k^2}\} \leq \frac{1}{k^2}.$$

Moreover,

$$\mathbb{P}\{t_k \geq 2/k^2\} \leq \mathbb{P}\{|t_k - 1/k^2| \geq \frac{1}{k^2}\} \leq \frac{1}{k^2}.$$

Borel-Cantelli then implies that $\{t_k \geq 2/k^2\}$ happens only finitely often almost surely, and we're done.

[10 pts] 6. Let $\mathbb{T} = \mathbb{N}$ and suppose F is a bounded predictable process with respect to filtration $\mathcal{F}$. Show that if M is a martingale with repect to $\mathcal{F}$ then so too is $X = \int F\, dM$.

Solution: By definition,

$$X_n = M_0 F_0 + (M_1 - M_0)F_1 + \cdots (M_n - M_{n-1})F_n.$$

To show that X_n is integrable, there exists $b > 0$ so that $|F_m| < b$ for all m. Hence

$$\begin{aligned} E[|X_n|] &\leq E[|M_0 F_0|] + E[|M_1 - M_0||F_1|] + \cdots E[|M_n - M_{n-1}||F_n|] \\ &\leq b(E[|M_0|] + E[|M_1 - M_0|] + \cdots + E[|M_n - M_{n-1}|]) \\ &\leq 2b(E[|M_0|] + E[|M_1|] + \cdots + E[|M_n|]) < \infty \end{aligned}$$

since each of the M_m are integrable.

Adaptability is obvious, since $M_0, \dots, M_n$ and $F_0, \dots, F_n$ are in $\mathcal{F}_n$.

Finally,

$$E_n[X_{n+1} - X_n] = E_n[F_{n+1}(M_{n+1} - M_n)] = F_{n+1}E_n[M_{n+1} - M_n] = 0.$$

[10 pts] 7. Let X be a martingale and ζ a stopping time. Show that $\widehat{X} = (X_{t\wedge\zeta})$ is a martingale.

Solution: If ζ is bounded, and $S \leq \zeta$, then $E_S[X_\zeta] = X_S$. In particular, $X_{t\wedge\zeta} = E_{t\wedge\zeta}[X_\zeta]$ and for $s < t$, $E_s[X_{t\wedge\zeta}] = E_s[E_{t\wedge\zeta}[X_\zeta]] = E_{s\wedge\zeta}[X_\zeta] = X_{s\wedge\zeta}$. Hence, in this situation, $\widehat{X}$ is a martingale. (Something about integrability)

For the general case, we will show that $\widehat{X}$ is a Doob martingale on $[0, b]$ for all b. Let T be a stopping time bounded by b. Then, $T \wedge \zeta$ is bounded by b, and $E[X_{T\wedge\zeta}] = E[X_0]$. Since $X_{T\wedge\zeta} = \widehat{X}_T$ and $X_0 = \widehat{X}_0$, we have $E[\widehat{X}_T] = E[\widehat{X}_0]$ for all stopping times T bounded by b. It follows that $\widehat{X}$ is a Doob martingale on $[0, b]$ for all b, and hence is a martingale.

8. Let $W = (W_t)$ be a standard Wiener process.

[5 pts] (a) Show that $W^2 - t$ is a martingale.

Solution: Since $E[W_t^2] = t$ we see that $W_t^2 - t$ is integrable. For the martingale condition, let $s < t$ and note

$$W_t^2 = (W_s + W_t - W_s)^2 = W_s^2 + 2W_s(W_t - W_s) + (W_t - W_s)^2$$

Thus,

$$E_s[W_t^2] = W_s^2 + 2W_sE_s[W_t - W_s] + E_s[(W_t - W_s)^2].$$

Since $W_t - W_s$ is independent of $\mathcal{F}_s$, we see $E_s[W_t - W_s] = E[W_t - W_s] = E[W_{t-s}] = 0$ and $E_s[(W_t - W_s)^2] = E[(W_{t-s})^2] = t - s$. Thus,

$$E_s[W_t^2] = W_s^2 + t - s$$

as desired.

[5 pts] (b) Let $T_a = \inf\{t > 0 : |W_t| \geq a\}$. Compute $E[T_a]$.

Solution: Assuming we can use the stopping time theorem, then

$$0 = E[W_0^2] = E[W_{T_a}^2 - T_a] = a^2 - E[T_a]$$

Question 8 continues...

and hence $E[T_a] = a^2$. The student should make some effort to justify why the stopping time theorem is applicable.

End of Exam