Instructions:

1. Read all questions carefully. If you are confused ask me!

2. You should have 6 pages including this page. Make sure you have the right number of pages.

3. Unless otherwise noted all conventions and notation follow that of Çınlar. If you are confused about notation, ask!

4. If necessary you may use the back of pages.

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1. Suppose \( X : (\Omega, \mathcal{H}) \to (E, \mathcal{E}) \) is a random variable taking values in the measurable space \((E, \mathcal{E})\). Show that \( V \in \sigma X \) iff there exists some measurable function \( f : (E, \mathcal{E}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \) such that \( V = f \circ X \).

**Solution:** If \( V = f \circ X \) then \( V \in \sigma X \).

\( \sigma X \) is generated by the \( p \)-system \( \mathcal{P} = \{X^{-1}(B) : B \subseteq \mathbb{R} \text{ Borel}\} \). Let \( H = X^{-1}(B) \) then \( 1_H = 1_{X^{-1}(B)} = 1_B \circ X \).

That is, if \( C = \{V \in \sigma X : V = f \circ X \text{ for some measurable } f : E \to \mathbb{R}\} \) then \( 1_H \in C \).

If \( V \) and \( W \) are in \( C \) then there exist measurable \( f \) and \( g \) such that \( V = f \circ X \) and \( W = g \circ X \). Thus \( cV + W = cf \circ X + g \circ X = (cf + g) \circ X \in C \).

Finally, suppose \( V_n \nearrow V \) for \( V_n \in C \). Then there exist measurable functions \( f_n \) so that \( V_n = f_n \circ X \). We may assume that \( f_n \) is identically 0 on the complement of \( X(\Omega) \) by multiplying by the appropriate indicator function if necessary. Define \( f \) on the image of \( X \) by

\[
 f(x) = \lim f_n(x) \quad x = X(\omega)
\]

If we take \( f \) to be identically 0 on the complement of \( X(\Omega) \) then \( f_n \not\nearrow f \) and hence \( f \) is measurable. It follows that \( V = f \circ X \) and thus \( C \) is a monotone class. Since \( C \) contains \( \mathcal{P} \) it contains all measurable functions in \( \sigma X \).

2. Suppose \( X_n \) are real-valued random variables with mean 0 and set \( S_n = X_1 + \cdots + X_n \). Show that for every \( a > 0 \),

\[
 \mathbb{P}\{\max_{k \leq n} |S_k| > a\} \leq \frac{E[S_n^2]}{a^2}.
\]

**Solution:** This is simply Kolmogorov’s inequality. Define the random variable \( N(\omega) = \inf\{k : |S_k(\omega)| > a\} \). Note that \( 1_{N=k} \) can be given in terms of the random variables \( X_1, \ldots, X_k \) and hence \( U := 1_{N=k}S_k \) depends only on the variables \( X_1, \ldots, X_k \) and is independent of \( V := S_n - S_k \). Since \( E[V] = 0 \), we have

\[
 0 = E[V]E[U] = E[VU] = E[S_k(S_n - S_k)1_{N=k}]
\]

Also, \( S_n^2 = [S_k + (S_n - S_k)]^2 \geq S_k^2 + 2(S_n - S_k)S_k \) and on the event \( \{N = k\} \), \( S_k^2 > a^2 \).

Thus,

\[
 E[S_n^21_{N=k}] \geq a^2E[1_{N=k}] + 2E[S_k(S_n - S_k)1_{N=k}] = a^2\mathbb{P}\{N = k\},
\]

and

\[
 a^2\mathbb{P}\{N \leq n\} \leq \sum_{k=1}^n E[S_n^21_{N=k}] = E[S_n^21_{N\leq n}] \leq E[S_n^2] = \text{Var}(S_n).
\]

This completes the proof since \( \{N \leq n\} = \{\max_{k \leq n} |S_k| > a\} \).
3. Suppose \((\mu_n)\) is a tight sequence of probability measures on \(\mathbb{R}\). Show that every subsequence of \((\mu_n)\) has a further subsequence that is weakly convergent. If you use Helly’s theorem, give the precise statement, though you need not prove it.

**Solution:** Let \((c_n)\) be the corresponding sequence of distribution functions of \((\mu_n)\). By Helly’s Theorem there is a subsequence \((c_{n'})\) and a distribution function \(c\) so that \(c_{n'}\) converges to \(c\) at every point of continuity of \(c\). Without tightness there is no guarantee that \(c\) is a distribution function in the sense of probability. That is, it remains to show

\[
\lim_{x \to -\infty} c(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} c(x) = 1
\]

By tightness, there exists a closed interval \([a,b]\) such that \(\mu_n[a,b] > 1 - \epsilon\) for all \(n\). Thus, \(\mu_n(-\infty,a) < \epsilon\) and \(\mu_n(-\infty,b] > 1 - \epsilon\). This implies that \(c(a) < \epsilon\) and \(c(b) > 1 - \epsilon\). Since \(\epsilon\) was arbitrary this shows that \(c\) is a distribution function in the sense of probability. If \(\mu\) is the corresponding probability measure on \(\mathbb{R}\) then \(\mu_{n'}\) converges weakly to \(\mu\).

4. Given probability space \((\Omega, \mathcal{H}, \mathbb{P})\), and sub-sigma algebra \(\mathcal{F} \subset \mathcal{H}\).

   (a) What does it mean for \(\mathbb{P}_\mathcal{F}\) to have a regular version? Be precise and define all relevant terms.

   **Solution:** \(Q(\omega, B)\) is a regular version for \(\mathbb{P}_\mathcal{F}\) if \(Q\) is a probability kernel from \((\Omega, \mathcal{H})\) into \((\Omega, \mathcal{F})\) so that

   \[
   E_\mathcal{F}[1_H](\omega) = Q(\omega, H).
   \]

   (b) Suppose \(\mathbb{P}_\mathcal{F}\) has a regular version \(Q\) and \(X\) is a real-valued random variable in \(L^1\). Show that a version of \(E_\mathcal{F}[X]\) is given by

   \[
   \omega \mapsto \int_\Omega X(\omega') Q_\omega(d\omega')
   \]

   **Solution:** Note that by definition,

   \[
   E_\mathcal{F}[1_H](\omega) = Q(\omega, H) = \int_\Omega 1_H(\omega') Q_\omega(d\omega')
   \]

   Monotone class, or definition of conditional expectation with monotone convergence does the rest.
5. Let $0 < T_1 < T_2 < \cdots$ be a sequence of random times and define the process $N = (N_t)$ by

$$N_t = \sum_{n=1}^{\infty} 1_{[0,t]} \circ T_n$$

Let $\mathcal{F}$ be the filtration generated by $N$.

(a) Show that $T_k$ is a stopping time of $\mathcal{F}$.

Solution: $\{T_k \leq t\} = \{N_t \geq k\} \in \mathcal{F}$

(b) Suppose $T_k = t_1 + \cdots + t_k$ for some independence of positive random variables $\{t_j\}$. Suppose $E[t_j] = 1/j^2$. Suppose $\text{Var}(t_j) = \frac{1}{j^6}$ and show that with probability 1 there is a random time $T$ such that $T$ is almost surely finite, and $N_T = +\infty$ almost surely.

Solution: Note that if $t_k < 2/k^2$ for all but finitely many then $T = \sum t_k$ is finite, and $N_T = \infty$. Thus it suffices to show that $t_k \geq 2/k^2$ only finitely often almost surely. By Chebyshev’s inequality, for $\sigma_k^2 = \text{Var}(t_k) = 1/k^6$

$$\mathbb{P}\{|t_k - 1/k^2| \geq c\sigma_k\} \leq \frac{1}{c^2}$$

With $c = k$ this becomes

$$\mathbb{P}\{|t_k - 1/k^2| \geq \frac{1}{k^2}\} \leq \frac{1}{k^2}.$$ 

Moreover,

$$\mathbb{P}\{t_k \geq 2/k^2\} \leq \mathbb{P}\{|t_k - 1/k^2| \geq \frac{1}{k^2}\} \leq \frac{1}{k^2}.$$ 

Borel-Cantelli then implies that $\{t_k \geq 2/k^2\}$ happens only finitely often almost surely, and we’re done.

6. Let $\mathbb{T} = \mathbb{N}$ and suppose $F$ is a bounded predictable process with respect to filtration $\mathcal{F}$. Show that if $M$ is a martingale with respect to $\mathcal{F}$ then so too is $X = \int F \, dM$.

Solution: By definition,

$$X_n = M_0F_0 + (M_1 - M_0)F_1 + \cdots (M_n - M_{n-1})F_n.$$ 

To show that $X_n$ is integrable, there exists $b > 0$ so that $|F_m| < b$ for all $m$. Hence

$$E[|X_n|] \leq E[|M_0F_0|] + E[|M_1 - M_0||F_1|] + \cdots E[|M_n - M_{n-1}||F_n|]$$

$$\leq b(E[|M_0|] + E[|M_1 - M_0|] + \cdots + E[|M_n - M_{n-1}|])$$

$$\leq 2b(E[|M_0|] + E[|M_1|] + \cdots + E[|M_n|]) < \infty.$$
since each of the $M_m$ are integrable.
Adaptability is obvious, since $M_0, \ldots, M_n$ and $F_0, \ldots, F_n$ are in $\mathcal{F}_n$.
Finally,
$$E_n[X_{n+1} - X_n] = E_n[F_{n+1}(M_{n+1} - M_n)] = F_{n+1}E_n[M_{n+1} - M_n] = 0.$$ 

7. Let $X$ be a martingale and $\zeta$ a stopping time. Show that $\hat{X} = (X_{t \wedge \zeta})$ is a martingale.

**Solution:** If $\zeta$ is bounded, and $S \leq \zeta$, then $E_S[X_\zeta] = X_S$. In particular, $X_{t \wedge \zeta} = E_{t \wedge \zeta}[X_\zeta]$ and for $s < t$, $E_s[X_{t \wedge \zeta}] = E_s[E_{t \wedge \zeta}[X_\zeta]] = E_{s \wedge \zeta}[X_\zeta] = X_{s \wedge \zeta}$. Hence, in this situation, $\hat{X}$ is a martingale. (Something about integrability)

For the general case, we will show that $\hat{X}$ is a Doob martingale on $[0, b]$ for all $b$. Let $T$ be a stopping time bounded by $b$. Then, $T \wedge \zeta$ is bounded by $b$, and $E[X_{T \wedge \zeta}] = E[X_0]$. Since $X_{T \wedge \zeta} = \hat{X}_T$ and $X_0 = \hat{X}_0$, we have $E[\hat{X}_T] = E[\hat{X}_0]$ for all stopping times $T$ bounded by $b$. It follows that $\hat{X}$ is a Doob martingale on $[0, b]$ for all $b$, and hence is a martingale.

8. Let $W = (W_t)$ be a standard Wiener process.

(a) Show that $W^2 - t$ is a martingale.

**Solution:** Since $E[W_t^2] = t$ we see that $W_t^2 - t$ is integrable. For the martingale condition, let $s < t$ and note
$$W_t^2 = (W_s + W_t - W_s)^2 = W_s^2 + 2W_s(W_t - W_s) + (W_t - W_s)^2$$
Thus,
$$E_s[W_t^2] = W_s^2 + 2W_sE_s[W_t - W_s] + E_s[(W_t - W_s)^2].$$
Since $W_t - W_s$ is independent of $\mathcal{F}_s$, we see $E_s[W_t - W_s] = E[W_t - W_s] = E[W_{t-s}] = 0$ and $E_s[(W_t - W_s)^2] = E[(W_{t-s})^2] = t - s$. Thus,
$$E_s[W_t^2] = W_s^2 + t - s$$
as desired.

(b) Let $T_a = \inf\{t > 0 : |W_t| \geq a\}$. Compute $E[T_a]$.

**Solution:** Assuming we can use the stopping time theorem, then
$$0 = E[W_0^2] = E[W_{T_a}^2 - T_a] = a^2 - E[T_a]$$

Question 8 continues...
and hence $E[T_a] = a^2$. The student should make some effort to justify why the stopping time theorem is applicable.