ANALYSIS QUALIFYING EXAM FALL 2017

Problem 1. Determine, with justification, the limit

$$\lim_{n\to\infty}\int_0^\infty \frac{1-\cos(nx)}{n^2x^2}\,dx.$$

Problem 2. Let $f \in L^1([0,\infty))$ (with respect to the usual Lebesgue measure). Suppose that f is uniformly continuous. Prove that $f \in C_0([0,\infty))$.

Problem 3. Let $p \in (1,\infty)$ and let $f_1, f_2, \ldots : \mathbb{R} \to \mathbb{C}$ be L^p functions. Assume that $\sum_{n=1}^\infty \|f_n\|_p^p < \infty$.

(1) Suppose that the sets

$$E_n = \big\{x \in \mathbb{R} \colon f_n(x) \neq 0\big\}$$

are disjoint. Prove that the series $\sum_{n=1}^\infty f_n$ converges in $L^p(\mathbb{R})$.

(2) Give an example to show that the conclusion of part (1) can fail if the sets E_n are not assumed disjoint.

Problem 4. Let E be a vector space over $\mathbb{C}$, and let $\langle\cdot,\cdot\rangle$ be a function from $E\times E$ to E such that:

(1) $\langle\eta,\xi\rangle = \overline{\langle\xi,\eta\rangle}$ for all $\xi,\eta \in E$.
(2) $\langle\xi_1+\xi_2,\,\eta\rangle = \langle\xi_1,\eta\rangle + \langle\xi_2,\eta\rangle$ for all $\xi_1,\xi_2,\eta \in E$.
(3) $\langle\alpha\xi,\eta = \alpha\langle\xi,\eta\rangle$ for all $\alpha \in \mathbb{C}$ and all $\xi,\eta \in E$.
(4) $\langle\xi,\xi\rangle \geq 0$ for all $\xi \in E$.

(This is the list of axioms for a scalar product, except that we have omitted the one which says $\langle\xi,\xi\rangle = 0$ implies $\xi = 0$.)

Prove that $N = \{\xi \in E \colon \langle\xi,\xi\rangle = 0\}$ is a vector subspasce of E, and that the formula $\langle\xi+N,\,\eta+N\rangle_0 = \langle\xi,\eta\rangle$, for $\xi,\eta \in E$, defines a scalar product on E/N.

(You may use the fact that the Schwarz inequality holds for $\langle\cdot,\cdot\rangle$.)

Problem 5. Prove that $l^1(\mathbb{Z})$ is not a Hilbert space. That is, prove that there does not exist any scalar product $\langle\cdot,\cdot\rangle$ on $l^1(\mathbb{Z})$ whose associated norm is the usual norm on $l^1(\mathbb{Z})$.

Hint: Find sone property of Hilbert spaces which $l^1(\mathbb{Z})$ does not have.

Problem 6. Let (X,μ) be a σ-finite measure space. Let $p \in [1,\infty)$. Let M be the set of measurable functions $f \colon X \to \mathbb{C}$ such that whenever $\xi \in L^p(X,\mu)$ then the function $x \mapsto (f\cdot\xi)(x) = f(x)\xi(x)$ is also in $L^p(X,\mu)$. For $f \in M$, let $m(f)\colon L^p(X,\mu) \to L^p(X,\mu)$ be the linear map defined by $(m(f)\xi)(x) = f(x)\xi(x)$ for $\xi \in L^p(X,\mu)$ and $x \in X$.

Suppose that $f \colon X \to \mathbb{C}$ is bounded and measurable, and assume that $m(f)$ is surjective. Prove that f is nonzero almost everywhere and that $1/f$ is essentially bounded.

Problem 7. Define $f\colon \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \int_0^1 e^{zt^3\cos(t)}\,dt$$

for $z \in \mathbb{C}$. Prove that f is an entire function.

Problem 8. Prove or disprove: There exists an entire function f such that $f\left(\frac{1}{n}\right)$ is real for all $n \in \mathbb{Z}_{>0}$, $f\left(\frac{1}{n}\right) > 0$ when n is even, and $f\left(\frac{1}{n}\right) < 0$ when n is odd.

Hint: Show that if f is such a function, then $f(z)$ is real for all real z.

Problem 9. In this problem, we use the notation $B_r(a) = \{z \in \mathbb{C} \colon |z| < r\}$, the open unit ball in $\mathbb{C}$ about a of radius r.

Let $\varepsilon > 0$, and let $f \colon B_{1+\varepsilon}(0) \to \mathbb{C}$ be a holomorphic function such that $|f(z)| = 1$ for all $z \in \mathbb{C}$ with $|z| = 1$, and such that $f(z) \neq 0$ for all $z \in B_1(0) \setminus \{0\}$. Prove that there are $n \in \mathbb{Z}_{\geq 0}$ and a constant α with $|\alpha| = 1$ such that $f(z) = \alpha z^n$ for all $z \in B_{1+\varepsilon}(0)$.