ANALYSIS QUALIFYING EXAM FALL 2017: SOLUTIONS

Problem 1. Determine, with justification, the limit

$$\lim_{n\to\infty} \int_0^\infty \frac{1-\cos(nx)}{n^2x^2}\,dx.$$

Solution. For an integer $n > 0$, define $g_n \colon (0,\infty) \to \mathbb{R}$ by

$$g_n(x) = \frac{1-\cos(nx)}{n^2x^2}.$$

Also define $g \colon (0,\infty) \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & x \in [0,1] \\ 2/x^2 & x \in (1,\infty). \end{cases}$$

We claim that the following three things are true:

(1) For $x \in (0,\infty)$, we have $\lim_{n\to\infty} g_n(x) = 0$.
(2) For $n > 0$ and $x \in (0,\infty)$, we have $0 \le g_n(x) \le g(x)$.
(3) g is integrable.

Combining these three facts with the Dominated Convergence Theorem (used at the second step), we get

$$\lim_{n\to\infty} \int_0^\infty \frac{1-\cos(nx)}{n^2x^2}\,dx = \lim_{n\to\infty} \int_0^\infty g_n(x)\,dx = \int_0^\infty \lim_{n\to\infty} g_n(x)\,dx = 0.$$

It remains to prove the three statements above. The statements (1) and (3) are obvious. For (2), the relation $0 \le g_n(x)$ is clear, since $-1 \le \cos(nx) \le 1$. For the other inequality, we first consider the case $x > 1$. Then $1 - \cos(nx) \le 2$, so

$$\frac{1-\cos(nx)}{n^2x^2} \le \frac{2}{n^2x^2} \le \frac{2}{x^2}.$$

Now suppose that $0 < x \le 1$. We start with the inequality $\sin(t) \le t$ for all $t \in [0,\infty)$. Integrating from 0 to x, we get $-\cos(x) + 1 \le \frac{1}{2}x^2$ for all $x \in [0,\infty)$. Therefore, for $x \in [0,\infty)$ and $n \in \mathbb{Z}_{>0}$, we have

$$1 - \cos(nx) \le \frac{1}{2}n^2x^2 \le n^2x^2,$$

from which it follows that $g_n(x) \le 1$. □

Alternate solution (sketch). Define $f \colon (0,\infty) \to \mathbb{R}$ by

$$f(x) = \frac{1-\cos(x)}{x^2}.$$

We are supposed to find

$$\lim_{n\to\infty} \int_0^\infty f(nx)\,dx.$$

Observe that $\lim_{x\to\infty} f(x) = 0$ and $f(x) \geq 0$ for all $x \in (0,\infty)$. Next, prove that f is bounded. Since f is continuous, it is enough to show that $\lim_{x\to 0^+} f(x)$ exists. This can be done by using power series to show that $1 - \cos(x)$ is holomorphic on a neighborhood of 0 and has a zero of order 2 there, or by using two applications of L'Hospital's Rule. (The limit is $\frac{1}{2}$.)

Now set $M = \sup_{x\in(0,\infty)} f(x)$. Define $g\colon (0,\infty) \to \mathbb{R}$ by

$$g(x) = \begin{cases} M & x \in [0,1] \\ 2/x^2 & x \in (1,\infty). \end{cases}$$

Prove as in the first solution that for $x \in (1,\infty)$,

$$f(nx) = \frac{1-\cos(nx)}{n^2x^2} \leq \frac{2}{n^2x^2} \leq \frac{2}{x^2} = g(x).$$

Also, for $x \in (0,1]$, we have $f(nx) \leq M = g(x)$.

We now use $\lim_{n\to\infty} f(nx) = 0$ for $x \in (0,\infty)$ and the Dominated Convergence Theorem (taking the dominating function to be g) at the second step to get

$$\lim_{n\to\infty} \int_0^\infty \frac{1-\cos(nx)}{n^2x^2}\,dx = \lim_{n\to\infty} \int_0^\infty f(nx)\,dx = \int_0^\infty \lim_{n\to\infty} f(nx)\,dx = 0.$$

This completes the solution. □

Second alternate solution (sketch). Use the methods of either of the first two solutions to show that the function $f\colon [0,\infty) \to \mathbb{R}$ given by

$$f(x) = \frac{1-\cos(x)}{x^2}\,dx$$

is nonnegative and integrable. Now, using the change of variables $y = nx$ at the second step, we have

$$\lim_{n\to\infty} \int_0^\infty \frac{1-\cos(nx)}{n^2x^2}\,dx = \lim_{n\to\infty} \int_0^\infty f(nx)\,dx = \lim_{n\to\infty} \frac{1}{n} \int_0^\infty f(y)\,dy = 0.$$

This completes the solution. □

Problem 2. Let $f \in L^1([0,\infty))$ (with respect to the usual Lebesgue measure). Suppose that f is uniformly continuous. Prove that $f \in C_0([0,\infty))$.

Solution. Suppose that $f \notin C_0([0,\infty))$. Then there exists $\varepsilon > 0$ such that for all $n \in \mathbb{Z}_{>0}$ there is $x_n \in [n,\infty)$ such that $|f(x_n)| > \varepsilon$. By passing to a subsequence of $(x_n)_{n\in\mathbb{Z}_{>0}}$, we may assume that $x_{n+1} > x_n + 1$ for all $n \in \mathbb{Z}_{>0}$.

Choose $\delta > 0$ such that for all $x, y \in [0,\infty)$ with $|x-y| < \delta$, we have $|f(x) - f(y)| < \frac{\varepsilon}{2}$, and also such that $\delta < \frac{1}{2}$. Define

$$E = \bigcup_{n=1}^\infty (x_n - \delta,\, x_n + \delta).$$

The sets in the union on the right are disjoint (since $\delta < \frac{1}{2}$ and $x_{n+1} > x_n + 1$), so E has infinite Lebesgue measure. For $x \in E$ there is $n \in \mathbb{Z}_{>0}$ such that $|x - x_n| < \delta$, so that

$$|f(x)| \geq |f(x_n)| - |f(x_n) - f(x)| > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Therefore $|f(x)| > \frac{\varepsilon}{2}$ for x in a set of infinite measure, and $\frac{\varepsilon}{2} > 0$, so $\int_{[0,\infty)} |f| = \infty$. ☐

Problem 3. Let $p \in (1, \infty)$ and let $f_1, f_2, \ldots \colon \mathbb{R} \to \mathbb{C}$ be L^p functions. Assume that $\sum_{n=1}^{\infty} \|f_n\|_p^p < \infty$.

(1) Suppose that the sets
$$E_n = \{x \in \mathbb{R} \colon f_n(x) \neq 0\}$$
are disjoint. Prove that the series $\sum_{n=1}^{\infty} f_n$ converges in $L^p(\mathbb{R})$.

(2) Give an example to show that the conclusion of part (1) can fail if the sets E_n are not assumed disjoint.

Solution. For part (1), recall the corollary to the Monotone Convergence Theorem: if (X, μ) is a measure space and $g_1, g_2, \ldots \colon X \to [0, \infty]$ are nonnegative measurable functions, then
$$\int_X \left(\sum_{n=1}^{\infty} g_n \right) d\mu = \sum_{n=1}^{\infty} \int_X g_n \, d\mu.$$

Define
$$f(x) = \begin{cases} f_n(x) & n \in \mathbb{Z}_{>0} \text{ and } x \in E_n \\ 0 & x \in \mathbb{R} \setminus \bigcup_{n=1}^{\infty} E_n. \end{cases}$$
Then for all $n \in \mathbb{Z}_{>0}$, using disjointness of the sets E_k at the second step and the corollary to the Monotone Convergence Theorem recalled above at the third step, we have
$$\left\| f - \sum_{k=1}^{n} f_k \right\|_p^p = \int_{\mathbb{R}} \left| \sum_{k=n+1}^{\infty} f_k(x) \right|^p dx = \int_{\mathbb{R}} \sum_{k=n+1}^{\infty} |f_k(x)|^p \, dx = \sum_{k=n+1}^{\infty} \|f_k\|_p^p.$$
Since $\sum_{n=1}^{\infty} \|f_n\|_p^p < \infty$, it follows that
$$\lim_{n \to \infty} \sum_{k=n+1}^{\infty} \|f_k\|_p^p = 0.$$
Therefore $\sum_{n=1}^{\infty} f_n$ converges in $L^p(\mathbb{R})$ to f.

For part (2), define $f_n = \frac{1}{n} \chi_{[0,1]}$ for $n \in \mathbb{Z}_{>0}$. Since $p > 1$,
$$\sum_{n=1}^{\infty} \|f_n\|_p^p = \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty.$$
Suppose $f \colon \mathbb{R} \to \mathbb{C}$ is an L^p function and $\sum_{n=1}^{\infty} f_n$ converges in $L^p(\mathbb{R})$ to f. Then there is a subsequence $(g_{l(n)})_{n \in \mathbb{Z}_{>0}}$ of the sequence of partial sums $g_n = \sum_{k=1}^{n} f_k$ such that $\lim_{n \to \infty} g_{l(n)}(x) = f(x)$ for almost all $x \in \mathbb{R}$. Since
$$\sum_{n=1}^{\infty} f_n(x) = \infty$$
for all $x \in [0,1]$, it follows that $f(x) = \infty$ for almost all $x \in [0,1]$. Since $[0,1]$ has strictly positive measure, this contradicts the assumption that f is an L^p function. ☐

Alternate solution to (1) (outline). At the beginning of the solution to part (1), instead of citing the corollary to the Monotone Convergence Theorem, derive it directly from the Monotone Convergence Theorem. This is easy. □

Problem 4. Let E be a vector space over $\mathbb{C}$, and let $\langle \cdot, \cdot \rangle$ be a function from $E \times E$ to E such that:

(1) $\langle \eta, \xi \rangle = \overline{\langle \xi, \eta \rangle}$ for all $\xi, \eta \in E$.
(2) $\langle \xi_1 + \xi_2, \, \eta \rangle = \langle \xi_1, \eta \rangle + \langle \xi_2, \eta \rangle$ for all $\xi_1, \xi_2, \eta \in E$.
(3) $\langle \alpha \xi, \eta = \alpha \langle \xi, \eta \rangle$ for all $\alpha \in \mathbb{C}$ and all $\xi, \eta \in E$.
(4) $\langle \xi, \xi \rangle \geq 0$ for all $\xi \in E$.

(This is the list of axioms for a scalar product, except that we have omitted the one which says $\langle \xi, \xi \rangle = 0$ implies $\xi = 0$.)

Prove that $N = \{ \xi \in E \colon \langle \xi, \xi \rangle = 0 \}$ is a vector subspasce of E, and that the formula $\langle \xi + N, \, \eta + N \rangle_0 = \langle \xi, \eta \rangle$, for $\xi, \eta \in E$, defines a scalar product on E/N.

(You may use the fact that the Schwarz inequality holds for $\langle \cdot, \cdot \rangle$.)

Solution. Define

$$M = \big\{ \xi \in E \colon \langle \xi, \eta \rangle = 0 \text{ for all } \eta \in E \big\}.$$

We first claim that $M = N$. (This is the main step.) It is clear that $M \subset N$. For the reverse inclusion, let $\xi \in N$. For all $\eta \in E$, the Schwarz inequality implies that

$$0 \leq |\langle \xi, \eta \rangle|^2 \leq \langle \xi, \xi \rangle \langle \eta, \eta \rangle = 0 \cdot \langle \eta, \eta \rangle = 0,$$

so $\xi \in M$. The claim is proved.

For $\eta \in E$ define $\omega_\eta \colon E \to \mathbb{C}$ by $\omega_\eta(\xi) = \langle \xi, \eta \rangle$ for $\xi \in E$. Then ω_η is linear. We have $M = \bigcap_{\eta \in E} \operatorname{Ker}(\omega_\eta)$, so M is a vector subspace. Thus N is a vector subspace.

For every $\eta \in E$, we have $N \subset \operatorname{Ker}(\omega_\eta)$, so ω_η descends to a linear map $\overline{\omega}_\eta \colon E/N \to \mathbb{C}$, determined by $\overline{\omega}_\eta(\xi + N) = \langle \xi, \eta \rangle$ for all $\xi \in E$. Moreover, if $\eta_1, \eta_2 \in E$ satisfy $\eta_1 - \eta_2 \in N$, then for all $\xi \in E$ we have

$$\overline{\omega}_{\eta_1}(\xi + N) - \overline{\omega}_{\eta_2}(\xi + N) = \langle \xi, \, \eta_1 - \eta_2 \rangle = \overline{\langle \eta_1 - \eta_2, \, \xi \rangle} = 0.$$

Thus the formula $\langle \xi + N, \, \eta + N \rangle_0 = \langle \xi, \eta \rangle$, for $\xi, \eta \in E$, is a well defined map $E/N \times E/N \to \mathbb{C}$. It is easily checked that $\langle \cdot, \cdot \rangle_0$ satisfies (1), (2), (3), and (4) on E/N. By construction, if $\langle \xi + N, \, \xi + N \rangle_0 = 0$ then $\xi \in N$, so $\xi + N = 0$. This completes the proof that $\langle \cdot, \cdot \rangle_0$ is a scalar product on E/N. □

Problem 5. Prove that $l^1(\mathbb{Z})$ is not a Hilbert space. That is, prove that there does not exist any scalar product $\langle \cdot, \cdot \rangle$ on $l^1(\mathbb{Z})$ whose associated norm is the usual norm on $l^1(\mathbb{Z})$.

Hint: Find sone property of Hilbert spaces which $l^1(\mathbb{Z})$ does not have.

Solution. The space $l^1(\mathbb{Z})$ is known to be separable. The Riesz Representation Theorem tells us that its dual is $l^\infty(\mathbb{Z})$, which is known not to be separable. However, the dual of a separable Hilbert space H is separable, since the map which send $\eta \in H$ to the linear functional $\xi \mapsto \langle \xi, \eta \rangle$ is an isometric bijection from H to H^*. □

Second solution (sketch). The condition for equality in the Schwarz Inequality implies that if H is a Hilbert space and $\xi, \eta \in H$ satisfy $\|\xi\| = \|\eta\| = 1$ and $\xi \neq \eta$, then $\|\xi + \eta\| < 2$. However, this property fails in $l^1(\mathbb{Z})$, as is easily seen by taking $\xi = \chi_{\{0\}}$ and $\eta = \chi_{\{1\}}$. □

Third solution. The parallelogram law, valid in any Hilbert space, fails in $l^1(\mathbb{Z})$. Take $\xi = \chi_{\{0\}}$ and $\eta = \chi_{\{0\}}$. Then

$$2\|\xi\|_1^2 + 2\|\eta\|_1^2 = 2 + 2 = 4,$$

but

$$\|\xi + \eta\|_1^2 + \|\xi - \eta\|_1^2 = 4 + 4 = 8,$$

which is not the same. □

Fourth solution. We recall that if H is a Hilbert space, $M \subset H$ is a closed subspace, and $\xi \in H$, then there is a unique $\eta \in M$ which minimizes $\|\xi - \eta\|$. In $l^1(\mathbb{Z})$ take

$$M = \left\{ \xi \in l^1(\mathbb{Z}) \colon \sum_{n \in \mathbb{Z}} \xi(n) = 0 \right\},$$

which is a closed subspace since it is the kernel of the bounded linear functional $\omega \in l^1(\mathbb{Z})^*$ given by $\omega(\xi) = \sum_{n \in \mathbb{Z}} \xi(n)$ for $\xi \in l^1(\mathbb{Z})$.

Take $\xi = \frac{1}{2}\chi_{\{0,1\}}$. Then $\|\xi\| = 1$. Since $\|\omega\| = 1$ and $\omega(\xi) = 1$, we clearly have $\operatorname{dist}(\xi, M) \geq 1$. Take $\eta_1 = \frac{1}{2}\big(\chi_{\{0\}} - \chi_{\{1\}}\big)$ and $\eta_2 = -\eta_1$. These are both in M, and easy calculations show that $\|\xi - \eta_1\|_1 = \|\xi - \eta_2\|_1 = 1$. Therefore there is more than one point in M which minimizes the distance to ξ. □

Fifth solution (sketch). We claim that if H is a nonzero Hilbert space and $\omega \in H^*$, then there is $\xi \in H$ such that $\|\xi\| = 1$ and $\omega(\xi) = \|\omega\|$. The claim is trivial if $\omega = 0$. Otherwise, choose $\xi_0 \in H$ such that $\omega(\eta) = \langle \eta, \xi_0 \rangle$ for all $\eta \in H$, and recall that $\|\omega\| = \|\xi_0\|$. Take $\xi = \|\xi_0\|^{-1}\xi_0$.

Define $\omega \in l^1(\mathbb{Z})^*$ by $\omega(\xi) = \sum_{n \in \mathbb{Z}} \big(1 - \frac{1}{n^2+1}\big)\xi(n)$. Then it is not hard to show that $\|\omega\| = 1$. With a bit more work, one can show that if $\xi \in l^1(\mathbb{Z})$ satisfies $\|\xi\| = 1$, then $|\omega(\xi)| < 1$. □

Problem 6. Let (X, μ) be a σ-finite measure space. Let $p \in [1, \infty)$. Let M be the set of measurable functions $f \colon X \to \mathbb{C}$ such that whenever $\xi \in L^p(X, \mu)$ then the function $x \mapsto (f \cdot \xi)(x) = f(x)\xi(x)$ is also in $L^p(X, \mu)$. For $f \in M$, let $m(f) \colon L^p(X, \mu) \to L^p(X, \mu)$ be the linear map defined by $(m(f)\xi)(x) = f(x)\xi(x)$ for $\xi \in L^p(X, \mu)$ and $x \in X$.

Suppose that $f \colon X \to \mathbb{C}$ is bounded and measurable, and assume that $m(f)$ is surjective. Prove that f is nonzero almost everywhere and that $1/f$ is essentially bounded.

Solution. We first claim that f is nonzero almost everywhere. Suppose that there is $E \subset X$ such that $\mu(E) > 0$ and f vanishes on E. Then $\mu(E) < \infty$, so $\chi_E \in L^p(X, \mu)$. However, for any $\xi \in L^p(X, \mu)$, the function $f \cdot \xi$ vanishes on E, so can't be equal almost everywhere to χ_E. This contradicts surjectivity of $m(f)$. The claim is proved.

Next, we claim that $m(f)$ is injective. Let $\xi \colon X \to \mathbb{C}$ be an L^p function such that $f \cdot \xi$ vanishes almost everywhere; we must show that ξ vanishes almost everywhere. Set $F = \{x \in X \colon \xi(x) \neq 0\}$. Then $f(x) = 0$ for almost every $x \in F$. Since $\mu\big(\{x \in X \colon f(x) = 0\}\big) = 0$, this implies that F is the union of two μ-null sets, hence μ-null. The claim is proved.

Since f is bounded, $m(f)$ is also bounded. The Open Mapping Theorem now implies that $m(f)$ is invertible. Clearly $m(f)^{-1} = m(1/f)$. It therefore remains to prove that if $g \colon X \to \mathbb{C}$ is measurable and $m(g)$ is bounded, then g is essentially bounded. So suppose that g is not essentially bounded. Let $M \in (0, \infty)$. Choose a measurable set $E \subset X$ such that $\mu(E) > 0$ and $|g(x)| > M^p$ for all $x \in E$. Then $\chi_E \in L^p(X, \mu)$ since μ is finite. Moreover, $\|\chi_E\|_p^p = \mu(E) > 0$ and

$$\|m(g)\chi_E\|_p^p = \int_E |g(x)| \, d\mu(x) \geq M^p \mu(E) = M^p \|\chi_E\|_p^p.$$

Thus $\|m(g)\| \geq M$. Since $M \in (0, \infty)$ is arbitrary, we have contradicted boundedness of $m(g)$. □

With more work (not needed here), one gets $\|m(g)\| = \|g\|_\infty$.

Alternate solution (outline). Suppose $1/f$ is not essentially bounded. Choose disjoint subsets $E_1, E_2, \ldots \subset X$, all with strictly positive measure, and such that $|f(x)| < \frac{1}{n}$ for all $n \in \mathbb{Z}_{>0}$ and $x \in E_n$. For $n \in \mathbb{Z}_{>0}$ define

$$\alpha_n = \frac{n^{-(p+1)/p}}{\mu(E_n)^{1/p}}.$$

Then define

$$\xi = \sum_{n=1}^{\infty} \alpha_n \chi_{E_n}.$$

Since $p > 0$, one checks that

$$\|\xi\|_p^p = \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} < \infty,$$

so $\xi \in L^p(X, \mu)$. However, if $\eta \in L^p(X, \mu)$ and $m(f)\eta = \xi$, then for all $n \in \mathbb{Z}_{>0}$ and $x \in E_n$ we have $|\eta(x)| > n\alpha_n$, so $p > 0$ implies

$$\|\eta\|_p^p \geq \sum_{n=1}^{\infty} n^p \alpha_n^p \mu(E_n) \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

This contradiction shows that $m(f)$ is not surjective. □

Problem 7. Define $f \colon \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \int_0^1 e^{zt^3 \cos(t)} \, dt$$

for $z \in \mathbb{C}$. Prove that f is an entire function.

Solution. We will combine Morera's Theorem and Fubini's Theorem.

We first check that h is continuous. It is easiest to use the Dominated Convergence Theorem (although this theorem is more powerful than is really needed). Let $(z_n)_{n\in\mathbb{Z}_{>0}}$ be any convergent sequence in $\mathbb{C}$. Let $z = \lim_{n\to\infty} z_n$. We show that $\lim_{n\to\infty} f(z_n) = f(z)$. Let $K \subset \mathbb{C}$ be a compact set which contains $\{z_n \colon n \in \mathbb{Z}_{>0}\}$. Set

$$c(t,z) = e^{zt^3\cos(t)}$$

for $t \in [0,1]$ and $z \in K$. Then c is a continuous function on the compact set $[0,1]\times K$. So there exists $M \in [0,\infty)$ such that $|c(t,z)| \leq M$ for all $(t,z) \in [0,1]\times K$. Moreover, $\lim_{n\to\infty} c(t,z_n) = c(t,z)$ for all $t \in [0,1]$. Using the constant function M as the dominating function, we apply the Dominated Convergence Theorem to conclude that $\lim_{n\to\infty} f(z_n) = f(z)$. This completes the proof that h is continuous.

We now use Morera's Theorem to show that f is holomorphic. Let Δ be a triangle in $\mathbb{C}$. We take $\partial\Delta$ to be parametrized as a piecewise C^1 curve $\gamma\colon [\alpha,\beta] \to \mathbb{C}$. In the following calculation, we provide the justification for the use of Fubini's Theorem at the second step afterwards, and the fourth step follows from Cauchy's Theorem for a triangle:

$$\begin{aligned}
\int_{\partial\Delta} f(z)\,dz &= \int_\alpha^\beta \left(\int_{-1}^1 \frac{\exp(t\gamma(s))}{1+t^2}\,dt \right) \gamma'(s)\,ds \\
&= \int_{-1}^1 \left(\int_\alpha^\beta \frac{\exp(t\gamma(s))}{1+t^2} \gamma'(s)\,ds \right) dt \\
&= \int_{-1}^1 \left(\int_{\partial\Delta} e^{zt^3\cos(t)}\,dz \right) dt = \int_{-1}^1 0\,dt = 0.
\end{aligned}$$

Fubini's Theorem is justified as follows. There is a finite set $S \subset [\alpha,\beta]$ such that the integrand as a function of (s,t) is continuous on $([\alpha,\beta]\setminus S)\times[0,1]$. Since $S\times[0,1]$ has measure zero, the integrand is measurable. By compactness of $[\alpha,\beta]\times[0,1]$, the quantity

$$M = \sup_{s\in[\alpha,\beta]} \sup_{t\in[0,1]} \left| \frac{\exp(t\gamma(s))}{1+t^2} \right|$$

is finite, and by the definition of a piecewise C^1 curve, the quantity

$$R = \sup_{s\in[\alpha,\beta]\setminus S} |\gamma'(s)|$$

is finite. Therefore the integrand is dominated in absolute value by the constant function with value MR. This function is integrable on $[\alpha,\beta]\times[0,1]$, so the hypotheses of Fubini's Theorem are satisfied.

We have shown that f is continuous and $\int_{\partial\Delta} f(z)\,dz = 0$ for every triangle Δ in $\mathbb{C}$. So h is holomorphic by Morera's Theorem. □

Problem 8. Prove or disprove: There exists an entire function f such that $f\left(\frac{1}{n}\right)$ is real for all $n \in \mathbb{Z}_{>0}$, $f\left(\frac{1}{n}\right) > 0$ when n is even, and $f\left(\frac{1}{n}\right) < 0$ when n is odd.

Hint: Show that if f is such a function, then $f(z)$ is real for all real z.

Solution. There is no such function. We argue by contradiction. Suppose there is such a function f. Out first order of business is to prove that $f(z)$ is real for all real z.

Define $g\colon \mathbb{C} \to \mathbb{C}$ by $g(z) = \overline{f(\overline{z})}$ for $z \in \mathbb{C}$. We first claim that g is also entire. Since f is entire, there are $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}_{\geq 0}$ such that $f(z) = \sum_{n=1}^{\infty} a_n z^n$ for all $z \in \mathbb{C}$. Then

$$g(z) = \overline{f(\overline{z})} = \sum_{n=1}^{\infty} \overline{a_n \overline{z}^n} = \sum_{n=1}^{\infty} \overline{a_n} z^n$$

for all $z \in \mathbb{C}$, so g is holomorphic. This proves the claim.

We have $f(z) = g(z)$ for all $z \in \left\{ \frac{1}{n} \colon n \in \mathbb{Z}_{>0} \right\}$, which is a set with a limit point in the domain. Therefore $f = g$. It follows that $f(z)$ is real for all $z \in \mathbb{R}$.

For all $n \in \mathbb{Z}_{>0}$, the Intermediate Value Theorem provides $z_n \in \left(\frac{1}{n+1}, \frac{1}{n} \right) \subset \mathbb{R}$ such that $f(z_n) = 0$. Then $\lim_{n\to\infty} z_n = 0$. Therefore f and the zero function are entire functions which agree on a set with a limit point. So $f = 0$. This contradicts $f(1) < 0$. ☐

Alternate solution. This solution differs only in the proof that $g(z) = \overline{f(\overline{z})}$ is entire.

To show this, let $z \in \mathbb{C}$. Then

$$\begin{aligned} \lim_{h\to 0} \frac{g(z+h) - g(z)}{h} &= \lim_{h\to 0} \frac{g(z+\overline{h}) - g(z)}{\overline{h}} = \lim_{h\to 0} \frac{\overline{f(\overline{z}+h)} - \overline{f(\overline{z})}}{\overline{h}} \\ &= \overline{\lim_{h\to 0} \frac{f(\overline{z}+h) - f(\overline{z})}{h}} = \overline{f'(\overline{z})}. \end{aligned}$$

In particular, this limit exists. So g is entire. ☐

Problem 9. In this problem, we use the notation $B_r(a) = \{z \in \mathbb{C} \colon |z| < r\}$, the open unit ball in $\mathbb{C}$ about a of radius r.

Let $\varepsilon > 0$, and let $f\colon B_{1+\varepsilon}(0) \to \mathbb{C}$ be a holomorphic function such that $|f(z)| = 1$ for all $z \in \mathbb{C}$ with $|z| = 1$, and such that $f(z) \neq 0$ for all $z \in B_1(0) \setminus \{0\}$. Prove that there are $n \in \mathbb{Z}_{\geq 0}$ and a constant α with $|\alpha| = 1$ such that $f(z) = \alpha z^n$ for all $z \in B_{1+\varepsilon}(0)$.

Solution. If f has a zero at 0, it is clearly isolated. Therefore there are $n \in \mathbb{Z}_{\geq 0}$ and a holomorphic function $h\colon B_{1+\varepsilon}(0) \to \mathbb{C}$ such that $f(z) = z^n h(z)$ for all $z \in B_{1+\varepsilon}(0)$ and such that $h(0) \neq 0$.

Clearly $|h(z)| = 1$ for all $z \in \mathbb{C}$ with $|z| = 1$. The Maximum Modulus Theorem therefore implies that $|h(z)| \leq 1$ for all $z \in \overline{B_1(0)}$. Also, $z^n h(z) \neq 0$ for all $z \in \overline{B_1(0)} \setminus \{0\}$, and $h(0) \neq 0$, so there is an open set $U \subset B_{1+\varepsilon}(0)$ such that $\overline{B_1(0)} \subset U$ and $h(z) \neq 0$ for all $z \in U$. The function $g(z) = h(z)^{-1}$ satisfies $|g(z)| = 1$ for all $z \in \mathbb{C}$ such that $|z| = 1$, so the Maximum Modulus Theorem implies that $|g(z)| \leq 1$ for all $z \in \overline{B_1(0)}$. So $|h(z)| \geq 1$ for all $z \in \overline{B_1(0)}$. Therefore $|h|$ has a local maximum at every point of $B_1(0)$. Thus h is constant by the condition for equality in the Maximum Modulus Theorem. Let α be this constant. Clearly $|\alpha| = 1$ and $f(z) = \alpha z^n$ for all $z \in B_{1+\varepsilon}(0)$. ☐