Your Name:

Conventions: all rings and algebras are assumed to be unital.

Part I. True or false? If true provide a brief explanation, if false provide a counterexample (10 points each):

1. Let $\mathcal{F}, \mathcal{G} : \textbf{A} \to \textbf{B}$ be isomorphic functors and $A_1, A_2$ be objects of $\textbf{A}$. Investigate each of the following for true/false:
   
   (i) $\mathcal{F}(A_1) = \mathcal{F}(A_2)$ if and only if $\mathcal{G}(A_1) = \mathcal{G}(A_2)$.
   
   (ii) $\mathcal{F}(A_1) \cong \mathcal{F}(A_2)$ if and only if $\mathcal{G}(A_1) \cong \mathcal{G}(A_2)$.

   (i) is false. Let $\mathbf{A} = \mathbf{B}$ be the category with two objects $A_1$ and $A_2$ and four morphisms $\text{id}_{A_1} : A_1 \to A_1, \text{id}_{A_2} : A_2 \to A_2, f : A_1 \to A_2$ and $g : A_2 \to A_1$, such that $f$ and $g$ are isomorphisms. Let $\mathcal{F} : \textbf{A} \to \textbf{A}$ be the functor which sends both $A_1$ and $A_2$ to $A_1$, and all morphisms to $\text{id}_{A_1}$. Then $\mathcal{F}$ is isomorphic to the identity functor on $\mathbf{A}$.

   (ii) is true. If $\alpha$ is an isomorphism from $\mathcal{F}$ to $\mathcal{G}$, and $\mathcal{F}(A_1) \cong \mathcal{F}(A_2)$, then $\mathcal{G}(A_1) \cong \mathcal{F}(A_1) \cong \mathcal{F}(A_2) \cong \mathcal{G}(A_2)$, where the first isomorphism is $\alpha^{-1}_{A_1}$ and the third isomorphism is $\alpha_{A_2}$. The converse is similar.
2. If $A$ is a commutative algebra over an algebraically closed field then all irreducible $A$-modules are 1-dimensional.

False. Consider the left regular module over the $\mathbb{C}$-algebra $\mathbb{C}(x)$. 
3. If $R$ is an artinian ring having no non-zero nilpotent elements then $R$ is a direct sum of division rings.

True. $R$ is artinian, so $J(R)$ is nilpotent. By assumption $J(R) = 0$. It follows that, $R$ is left semisimple. by Wedderburn-Artin, $R$ is a direct sum of matrix rings over division rings. Any matrix ring of size greater than 1 has nilpotent elements. Hence $R$ is a direct sum of division rings.
4. Let $\varphi : X \to Y$ be a morphism of affine algebraic sets. Then $\varphi$ is surjective if and only if $\varphi^*$ is injective (recall that $\varphi^* : \mathbb{F}[Y] \to \mathbb{F}[X], f \mapsto f \circ \varphi$).

*False.* Note that
\[ \mathcal{I}(\text{im} \, \varphi) = \{ g \in \mathbb{F}[Y] \mid g(\varphi(x)) = 0 \text{ for any } x \in X \} = \{ g \in \mathbb{F}[Y] \mid \varphi^*(g) = 0 \} = \ker \varphi^*. \]

Since $\mathcal{V}(\mathcal{I}(\text{im} \, \varphi)) = \text{im} \varphi$, we conclude that $\varphi^*$ is injective if and only if the image of $\varphi$ is dense, which is possible without $\varphi$ being surjective, for example consider $\varphi : \mathcal{V}(xy - 1) \to \mathbb{C}, (x, y) \mapsto x$. 
5. If $R, R'$ are rings, $\mathcal{F} : R\text{-Mod} \to R'\text{-Mod}$ is a functor left adjoint to a functor $\mathcal{G} : R'\text{-Mod} \to R\text{-Mod}$, and $P$ is a projective $R$-module, then $\mathcal{F}P$ is a projective $R'$-module.

*False* without the assumption that $\mathcal{G}$ is exact. For example tensoring with $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$ does not send projectives to projectives, even though this functor is left adjoint to $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, -)$. 
Part II. Prove the following statements (10 points each):

1. Prove for any vector spaces $U$ and $V$ we have an isomorphism of vector spaces
   \[ \text{Hom}_F(U, V^*) \rightarrow \text{Hom}_F(V, U^*), \quad f \mapsto f^* \circ \iota_V, \]
   where $\iota_V : V \rightarrow V^{**}$ is the natural embedding.

   **Solution.** We consider the linear map $\iota_V : V \rightarrow V^{**}$, $v \mapsto \varphi_v$, where $\varphi_v(f) = f(v)$ for any $f \in V^*$. Consider the linear maps
   \[ \alpha : \text{Hom}_F(U, V^*) \rightarrow \text{Hom}_F(V, U^*), \quad f \mapsto f^* \circ \iota_V \]
   and
   \[ \beta : \text{Hom}_F(V, U^*) \rightarrow \text{Hom}_F(U, V^*), \quad g \mapsto g^* \circ \iota_U. \]
   It suffices to prove that $\alpha$ and $\beta$ are mutual inverses. We show that $\beta \circ \alpha = \text{id}$, the identity $\beta \circ \alpha = \text{id}$ follows by symmetry. We have
   \[
   \beta(\alpha(f)) = \alpha(f)^* \circ \iota_U \\
   = (f^* \circ \iota_V)^* \circ \iota_U \\
   = \iota_V^* \circ f^{**} \circ \iota_U,
   \]
   and to remain to prove that $\iota_V^* \circ f^{**} \circ \iota_U = f$. Well,
   \[
   (\iota_V^* \circ f^{**} \circ \iota_U)(u) = (f^* \circ \iota_V)^*(\iota_U(u)) \\
   = (f^* \circ \iota_V)^*(\varphi_u) \\
   = \varphi_u \circ f^* \circ \iota_V,
   \]
   so we have to prove that $\varphi_u \circ f^* \circ \iota_V = f(u)(v)$. Well,
   \[
   \varphi_u(f^*(\iota_V(v))) = \varphi_u(\iota_V(v) \circ f) = \iota_V(v)(f(u)) = f(u)(v). \]
2. Let $N$ be a normal subgroup of a finite group $G$ and $P$ be a Sylow $p$-subgroup of $G$. Then $P \cap N$ is a Sylow $p$-subgroup of $N$, and every Sylow $p$-subgroup of $N$ is of this form.

Solution. Let $S$ be a Sylow $p$-subgroup of $N$. There exists a Sylow $p$-subgroup $Q$ of $G$ which contains $S$. Now $Q \cap N \supseteq S$ implies $Q \cap N = S$. Thus every Sylow $p$-subgroup of $N$ is of the required form.

Furthermore, let $P$ be any Sylow $p$-subgroup of $G$. Since $Q$ and $P$ are conjugate in $G$, and $N$ is normal, $Q \cap H$ and $P \cap H$ have the same order. So $P \cap H$ is a Sylow $p$-subgroup of $N$. 
3. Let $V$ be an $R$-module. A family $(V_i)_{i \in I}$ of submodules of $V$ is called a directed system of submodules if for any $i, j \in I$ there exists $k \in I$ such that $V_i \subseteq V_k$ and $V_j \subseteq V_k$. Prove that $V$ is finitely generated if and only if the union $\bigcup_{i \in I} V_i$ of any directed set of proper submodules is proper. Deduce that a finitely generated module has a maximal proper submodule.

Solution. If $V$ is infinitely generated, then the set of all finitely generated submodules of $V$ is a directed set of proper submodules! The union is of course not proper, as every vector lies in a finitely generated submodule.

Conversely, if $V$ is finitely generated, and we have $(V_i)$ a directed set of proper submodules whose union is $V$, then we get a contradiction, as each of the finitely many generators belongs to some $V_i$, hence $V_{i_1} + \cdots + V_{i_n} = V$, which is impossible for a directed set of proper submodules.

The final statement now comes from the Zorn Lemma.
4. An element $g$ of a finite group $G$ is conjugate to $g^{-1}$ if and only if $\chi(g)$ is a real number for every character $\chi$.

The irreducible characters form a basis for $C(G)$, as do the indicator functions of the conjugacy classes of $G$. Hence $g$ and $h$ lie in the same conjugacy class of $G$ if and only if $\chi(g) = \chi(h)$ for all irreducible characters $\chi$. Hence $g$ and $g^{-1}$ lie in the same conjugacy class of $G$ if and only if $\chi(g) = \chi(g^{-1}) = \chi(g)$ for all irreducible characters $\chi$ if and only if $\chi(g) = \chi(g)$ for all characters $\chi$. 
5. Let $G$ be a finite group of automorphisms of the ring $A$, and $R = A^G = \{a \in A \mid ga = a \text{ for all } g \in G\}$ be the subring of invariants. Then the ring extension $R \subseteq A$ is integral.

Solution. For any $f \in A$, the coefficients of the monic polynomial $P_f(t) := \prod_{g \in G}(t - g \cdot f)$ belong to $A^G$. On the other hand $P_f(f) = 0$. 