ALGEBRA QUALIFYING EXAM, FALL 2017: SOLUTIONS

Your Name:

Conventions: all rings and algebras are assumed to be unital.

Part I. True or false? If true provide a brief explanation, if false provide a counterexample (10 points each):

1. Let $\mathcal{F}, \mathcal{G} : \mathbf{A} \to \mathbf{B}$ be isomorphic functors and A_1, A_2 be objects of $\mathbf{A}$. Investigate each of the following for true/false:
(i) $\mathcal{F}(A_1) = \mathcal{F}(A_2)$ if and only if $\mathcal{G}(A_1) = \mathcal{G}(A_2)$.
(ii) $\mathcal{F}(A_1) \cong \mathcal{F}(A_2)$ if and only if $\mathcal{G}(A_1) \cong \mathcal{G}(A_2)$.

(i) is *false.* Let $\mathbf{A} = \mathbf{B}$ be the category with two objects A_1 and A_2 and four morphisms $\mathrm{id}_{A_1} : A_1 \to A_1, \mathrm{id}_{A_2} : A_2 \to A_2, f : A_1 \to A_2$ and $g : A_2 \to A_1$, such that f and g are isomorphisms. Let $\mathcal{F} : \mathbf{A} \to \mathbf{A}$ be the functor which sends both A_1 and A_2 to A_1, and all morphisms to id_{A_1}. Then $\mathcal{F}$ is isomorphic to the identity functor on $\mathbf{A}$.

(ii) is *true.* If α is an isomorphism from $\mathcal{F}$ to $\mathcal{G}$, and $\mathcal{F}(A_1) \cong \mathcal{F}(A_2)$, then $\mathcal{G}(A_1) \cong \mathcal{F}(A_1) \cong \mathcal{F}(A_2) \cong \mathcal{G}(A_2)$, where the first isomorphism is $\alpha_{A_1}^{-1}$ and the third isomorphism is α_{A_2}. The converse is similar.

2. If A is a commutative algebra over an algebraically closed field then all irreducible A-modules are 1-dimensional.

False. Consider the left regular module over the $\mathbb{C}$-algebra $\mathbb{C}(x)$.

3. If R is an artinian ring having no non-zero nilpotent elements then R is a direct sum of division rings.

True. R is artinian, so $J(R)$ is nilpotent. By assumption $J(R) = 0$. It follows that, R is left semisimple. by Wedderburn-Artin, R is a direct sum of matrix rings over division rings. Any matrix ring of size greater than 1 has nilpotent elements. Hence R is a direct sum of division rings.

4. Let $\varphi : X \to Y$ be a morphism of affine algebraic sets. Then φ is surjective if and only if φ^* is injective (recall that $\varphi^* : \mathbb{F}[Y] \to \mathbb{F}[X], f \mapsto f \circ \varphi$).

False. Note that

$$\begin{aligned} \mathcal{I}(\operatorname{im}\varphi) &= \{g \in \mathbb{F}[Y] \mid g(\varphi(x)) = 0 \text{ for any } x \in X\} \\ &= \{g \in \mathbb{F}[Y] \mid \varphi^*(g) = 0\} = \ker \varphi^*. \end{aligned}$$

Since $\mathcal{V}(\mathcal{I}(\operatorname{im}\varphi)) = \overline{\operatorname{im}\varphi}$, we conclude that φ^* is injective if and only if the image of φ is dense, which is possible without φ being surjective, for example consider $\varphi : \mathcal{V}(xy-1) \to \mathbb{C},\ (x, y) \mapsto x$.

5. If R, R' are rings, $\mathcal{F} : R\text{-}\mathtt{Mod} \to R'\text{-}\mathtt{Mod}$ is a functor left adjoint to a functor $\mathcal{G} : R'\text{-}\mathtt{Mod} \to R\text{-}\mathtt{Mod}$, and P is a projective R-module, then $\mathcal{F}P$ is a projective R'-module.

False without the assumption that $\mathcal{G}$ is exact. For example tensoring with $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$ does not send projectives to projectives, even though this functor is left adjoint to $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, -)$.

Part II. Prove the following statements (10 points each):

1. Prove for *any* vector spaces U and V we have an isomorphism of vector spaces
$$\mathrm{Hom}_{\mathbb{F}}(U, V^*) \to \mathrm{Hom}_{\mathbb{F}}(V, U^*),\ f \mapsto f^* \circ \iota_V,$$
where $\iota_V : V \to V^{**}$ is the natural embedding.

Solution. We consider the linear map $\iota_V : V \to V^{**}, v \mapsto \varphi_v$, where $\varphi_v(f) = f(v)$ for any $f \in V^*$. Consider the linear maps
$$\alpha : \mathrm{Hom}_{\mathbb{F}}(U, V^*) \to \mathrm{Hom}_{\mathbb{F}}(V, U^*), \qquad f \mapsto f^* \circ \iota_V$$
and
$$\beta : \mathrm{Hom}_{\mathbb{F}}(V, U^*) \to \mathrm{Hom}_{\mathbb{F}}(U, V^*), \qquad g \mapsto g^* \circ \iota_U.$$
It suffices to prove that α and β are mutual inverses. We show that $\beta \circ \alpha = \mathrm{id}$, the identity $\beta \circ \alpha = \mathrm{id}$ follows by symmetry. We have
$$\begin{aligned}\beta(\alpha(f)) &= \alpha(f)^* \circ \iota_U \\ &= (f^* \circ \iota_V)^* \circ \iota_U \\ &= \iota_V^* \circ f^{**} \circ \iota_U,\end{aligned}$$
and to remains to prove that $\iota_V^* \circ f^{**} \circ \iota_U = f$. Well,
$$\begin{aligned}(\iota_V^* \circ f^{**} \circ \iota_U)(u) &= (f^* \circ \iota_V)^*(\iota_U(u)) \\ &= (f^* \circ \iota_V)^*(\varphi_u) \\ &= \varphi_u \circ f^* \circ \iota_V,\end{aligned}$$
so we have to prove that $\varphi_u \circ f^* \circ \iota_V = f(u)(v)$. Well,
$$\varphi_u(f^*(\iota_V(v))) = \varphi_u(\iota_V(v) \circ f) = \iota_V(v)(f(u)) = f(u)(v).$$

2. Let N be a normal subgroup of a finite group G and P be a Sylow p-subgroup of G. Then $P \cap N$ is a Sylow p-subgroup of N, and every Sylow p-subgroup of N is of this form.

Solution. Let S be a Sylow p-subgroup of N. There exists a Sylow p-subgroup Q of G which contains S. Now $Q \cap N \supseteq S$ implies $Q \cap N = S$. Thus every Sylow p-subgroup of N is of the required form.

Furthermore, let P be any Sylow p-subgroup of G. Since Q and P are conjugate in G, and N is normal, $Q \cap H$ and $P \cap H$ have the same order. So $P \cap H$ is a Sylow p-subgroup of N.

3. Let V be an R-module. A family $(V_i)_{i\in I}$ of submodules of V is called a *directed system of submodules* if for any $i, j \in I$ there exists $k \in I$ such that $V_i \subseteq V_k$ and $V_j \subseteq V_k$. Prove that V is finitely generated if and only if the union $\cup_{i\in I} V_i$ of any directed set of proper submodules is proper. Deduce that a finitely generated module has a maximal proper submodule.

Solution. If V is infinitely generated, then the set of all finitely generated submodules of V is a directed set of proper submodules! The union is of course not proper, as every vector lies in a finitely generated submodule.

Conversely, if V is finitely generated, and we have (V_i) a directed set of proper submodules whose union is V, then we get a contradiction, as each of the finitely many generators belongs to some V_i, hence $V_{i_1} + \cdots + V_{i_n} = V$, which is impossible for a *directed* set of *proper* submodules.

The final statement now comes from the Zorn Lemma.

4. An element g of a finite group G is conjugate to g^{-1} if and only if $\chi(g)$ is a real number for every character χ.

The irreducible characters form a basis for $C(G)$, as do the indicator functions of the conjugacy classes of G. Hence g and h lie in the same conjugacy class of G if and only if $\chi(g) = \chi(h)$ for all irreducible characters χ. Hence g and g^{-1} lie in the same conjugacy class of G if and only if $\chi(g) = \chi(g^{-1}) = \overline{\chi(g)}$ for all irreducible characters χ if and only if $\chi(g) = \overline{\chi(g)}$ for all characters χ.

5. Let G be a finite group of automorphisms of the ring A, and $R = A^G = \{a \in A \mid ga = a \text{ for all } g \in G\}$ be the subring of invariants. Then the ring extension $R \subseteq A$ is integral.

Solution. For any $f \in A$, the coefficients of the monic polynomial $P_f(t) := \prod_{g \in G}(t - g \cdot f)$ belong to A^G. On the other hand $P_f(f) = 0$.