# ALGEBRA QUALIFYING EXAM, FALL 2017

**Your Name:**

**Conventions: all rings and algebras are assumed to be unital.**

**Part I. True or false? If true provide a brief explanation, if false provide a counterexample (10 points each):**

1. Let $\mathcal{F}, \mathcal{G} : \mathbf{A} \to \mathbf{B}$ be isomorphic functors and $A_1, A_2$ be objects of $\mathbf{A}$. Investigate each of the following for true/false:
   (i) $\mathcal{F}(A_1) = \mathcal{F}(A_2)$ if and only if $\mathcal{G}(A_1) = \mathcal{G}(A_2)$.
   (ii) $\mathcal{F}(A_1) \cong \mathcal{F}(A_2)$ if and only if $\mathcal{G}(A_1) \cong \mathcal{G}(A_2)$.
2. If $A$ is a commutative algebra over an algebraically closed field then all irreducible $A$-modules are 1-dimensional.
3. If $R$ is an artinian ring having no non-zero nilpotent elements then $R$ is a direct sum of division rings.
4. Let $\varphi : X \to Y$ be a morphism of affine algebraic sets. Then $\varphi$ is surjective if and only if $\varphi^*$ is injective (recall that $\varphi^* : \mathbb{F}[Y] \to \mathbb{F}[X], f \mapsto f \circ \varphi$).
5. If $R, R'$ are rings, $\mathcal{F} : R\text{-}\mathtt{Mod} \to R'\text{-}\mathtt{Mod}$ is a functor left adjoint to a functor $\mathcal{G} : R'\text{-}\mathtt{Mod} \to R\text{-}\mathtt{Mod}$, and $P$ is a projective $R$-module, then $\mathcal{F}P$ is a projective $R'$-module.

**Part II. Prove the following statements (10 points each):**

1. Prove for *any* vector spaces $U$ and $V$ we have an isomorphism of vector spaces
$$\mathrm{Hom}_{\mathbb{F}}(U, V^*) \to \mathrm{Hom}_{\mathbb{F}}(V, U^*),\ f \mapsto f^* \circ \iota_V,$$
where $\iota_V : V \to V^{**}$ is the natural embedding.
2. Let $N$ be a normal subgroup of a finite group $G$ and $P$ be a Sylow $p$-subgroup of $G$. Then $P \cap N$ is a Sylow $p$-subgroup of $N$, and every Sylow $p$-subgroup of $N$ is of this form.
3. Let $V$ be an $R$-module. A family $(V_i)_{i \in I}$ of submodules of $V$ is called a *directed system of submodules* if for any $i, j \in I$ there exists $k \in I$ such that $V_i \subseteq V_k$ and $V_j \subseteq V_k$. Prove that $V$ is finitely generated if and only if the union $\cup_{i \in I} V_i$ of any

directed set of proper submodules is proper. Deduce that a finitely generated module has a maximal proper submodule.

4. An element $g$ of a finite group $G$ is conjugate to $g^{-1}$ if and only if $\chi(g)$ is a real number for every character $\chi$ of $G$.
5. Let $G$ be a finite group of automorphisms of the ring $A$, and $R = A^G = \{a \in A \mid ga = a \text{ for all } g \in G\}$ be the subring of invariants. Then the ring extension $R \subseteq A$ is integral.