Reflexive Polygons and Loops

by

Dongmin Roh

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A reflexive polytope is a special type of convex lattice polytope that only contains the origin as its interior lattice points. Its dual is also reflexive; they form a mirror pair. The Ehrhart series of a reflexive polytope exhibits a symmetric behavior as well. In this paper, we focus on the case when the dimension is 2. Reflexive polygons have a fascinating property that the number of boundary points of itself and its dual always sum up to 12. Furthermore, we extend this idea to a notion of reflexive polygons of higher index, called l-reflexive polygons, and generalize the case to dropping the condition of the convexivity of polygons, known as l-reflexive loops. We will observe that the magical 'number 12' extends to l-reflexive polygons and l-reflexive loops. We will look at how a given l-reflexive can be thought of as a 1-reflexive in the lattice generated by its boundary lattice points. Lastly, we observe that the roots of the Ehrhart Polynomial having real part equal to $-1/2$ imply that the corresponding polytope is reflexive.

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REFLEXIVE POLYGONS AND LOOPS

1. REFLEXIVE POLYGONS

1.1. What is a Reflexive Polygon?

It is crucial to first understand what *lattice* and *dual lattice* of integer points are. Then we define *lattice polytope* and *dual polytope* for any arbitrary dimension n.

Definition 1.1.0.1 (Lattice)**.** *A set in $\mathbb{R}^n$ is called a **lattice** if it is isomorphic to $\mathbb{Z}^n$.*

Definition 1.1.0.2 (Linear Functional)**.** *Let N be a vector space. A function $\psi : N \to \mathbb{Z}$ is a **linear functional** if $\psi(av + bw) = a\psi(v) + b\psi(w)$, $\forall a, b \in \mathbb{Z}$ and $\forall v, w \in N$.*

With Definition 1.1.0.2 in our hands, now we can define the *dual lattice* as the following:

Definition 1.1.0.3 (Dual Lattice)**.** *Let $N \cong \mathbb{Z}^n$ be a lattice. Then a **dual lattice** M of N is the set of all linear functionals from N to $\mathbb{Z}$.*

In general, given some lattice Λ, we are familiar with the definition of its *dual lattice* to be

$$\Lambda^* = \{y \in span(\Lambda) : x \in \Lambda, \langle x, y\rangle \in \mathbb{Z}\}$$

When a lattice $N = \mathbb{Z}^n$ its dual lattice M is again $\mathbb{Z}^n$; $\mathbb{Z}^n$ is self-dual.

Definition 1.1.0.4 (Lattice Polytope)**.** *A finite convex polytope $P \subseteq N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ of dimension n is called a **lattice polytope** if it has its vertices in the lattice $N \cong \mathbb{Z}^n$.*

Definition 1.1.0.5 (Dual Polytope)**.** *A **dual polytope** of P is defined as $P^* := \{y \in \mathbb{R}^n : \langle y, x\rangle \le 1, x \in P\}$.*

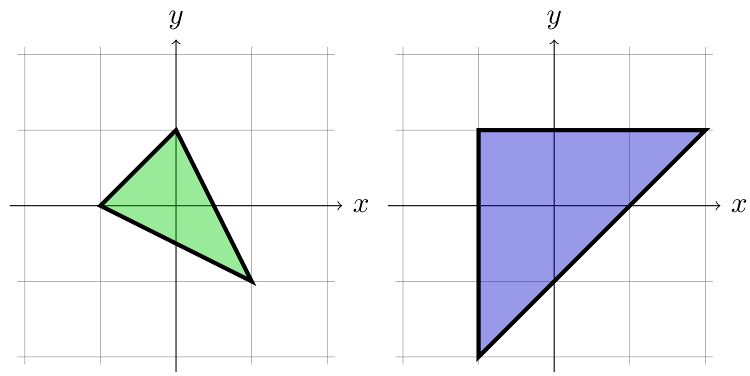
Depending on the authors the *dual polytope* is sometimes defined as $P^* := \{y \in \mathbb{R}^n : \langle y, x\rangle \ge -1, x \in P\}$, but we will see later on that both definitions suffice for our purpose.

Now we are finally well-suited to define a *reflexive polytope.*

Definition 1.1.0.6 (Reflexive Polytope)**.** *A lattice polytope P is* ***reflexive*** *if the origin is its only interior lattice point.*

What makes a *reflexive polytope* special is that its *dual polytope* is a *reflexive* lattice polytope as well. In fact, the definition of a *reflexive polytope* can be equivalently defined as: P is reflexive if and only if both P and its dual P^* are lattice polytopes.

Example 1.1.0.1. *Let's take a look at an example of a reflexive polygon P and its dual polygon P^* to get a better understanding of them.*

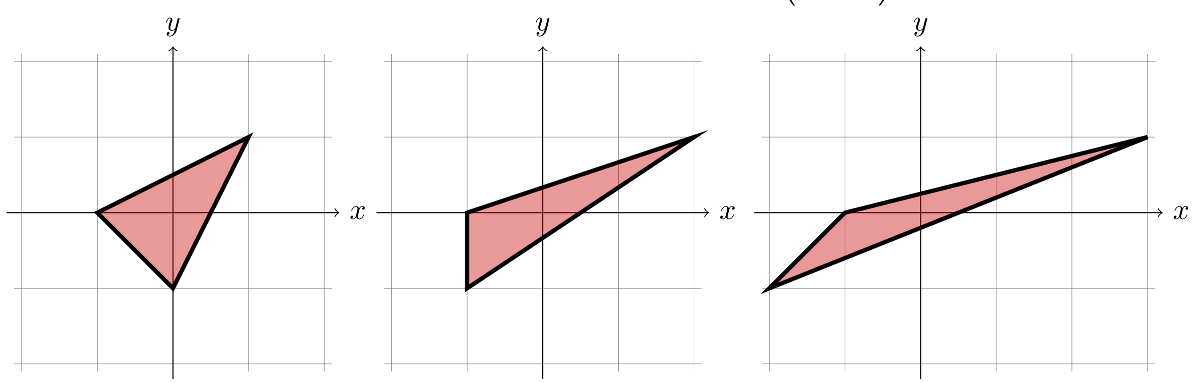


Given a reflexive polygon(on the left) we compute its dual in the following way. Let $\mathbf{y} = (x, y) \in \mathbb{R}^n$. Vertices of a given reflexive polygon are $(0, 1)$, $(-1, 0)$, and $(1, -1)$ going counterclockwise. Then, using the Definition 1.1.0.5 we obtain half-spaces $y \leq 1$, $-x \leq 1$, and $x - y \leq 1$ whose intersecting space form a dual polygon(on the right). One can easily check that $(P^*)^* = P$ as well.

Actually, there's a quicker way to compute the dual. The vertices of the dual can be found by computing $x_{i+1} - x_i$, where x_i's are boundary lattice points of the original reflexive polygon, and we take the convention of going counter-clockwise. Then rotate $x_{i+1} - x_i$ by 90 degrees clockwise, or left multiply by a matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The dual is simply the convex hull of the $x_{i+1} - x_i$. So for example, from the example above, the corresponding $x_{i+1} - x_i$ are $(-1, 2)$, $(-1, -1)$, and $(2, -1)$, then rotate them to get $(2, 1)$, $(-1, 1)$, and $(-1, -2)$, which are the vertices of the dual as desired. However, this simplicative method works for reflexive polygons only. The method works because the dual of a reflexive polygon is a reflexive polygon as well.

1.2. Classifying Reflexive Polygons

It isn't hard to convince oneself that there are infinitely many *reflexive polygons*. For example, consider the *reflexive polygon* from Example 1.1.0.1. We can create new *reflexive polygons* from this by 90 degree rotations or reflection across the x-axis or y-axis. A less obvious way is to apply a shear transformation by a matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.



The above drawings show the effects of the shear map performed three times. Notice that there is still only one lattice point in the interior and the vertices are lattice points after each shear transformation. Thus, iteratively applying the shear map will give us an infinite number of *reflexive polygons*.

But it is uninteresting and unintuitive to consider infinite number of *reflexive polygons*. Instead we will clasify each *reflexive polygons* to form equivalence classes, such that two *reflexive polygons* belong to the same equivalence class if there exists a reflection, a rotation, or a shear map between them. In other words, these transformations are 2x2 matrices that send integer points to integer points and the plane to itself, so the matrices are invertible whose determinants are either 1 or -1.

It turns out that these matrices form a very well known group.

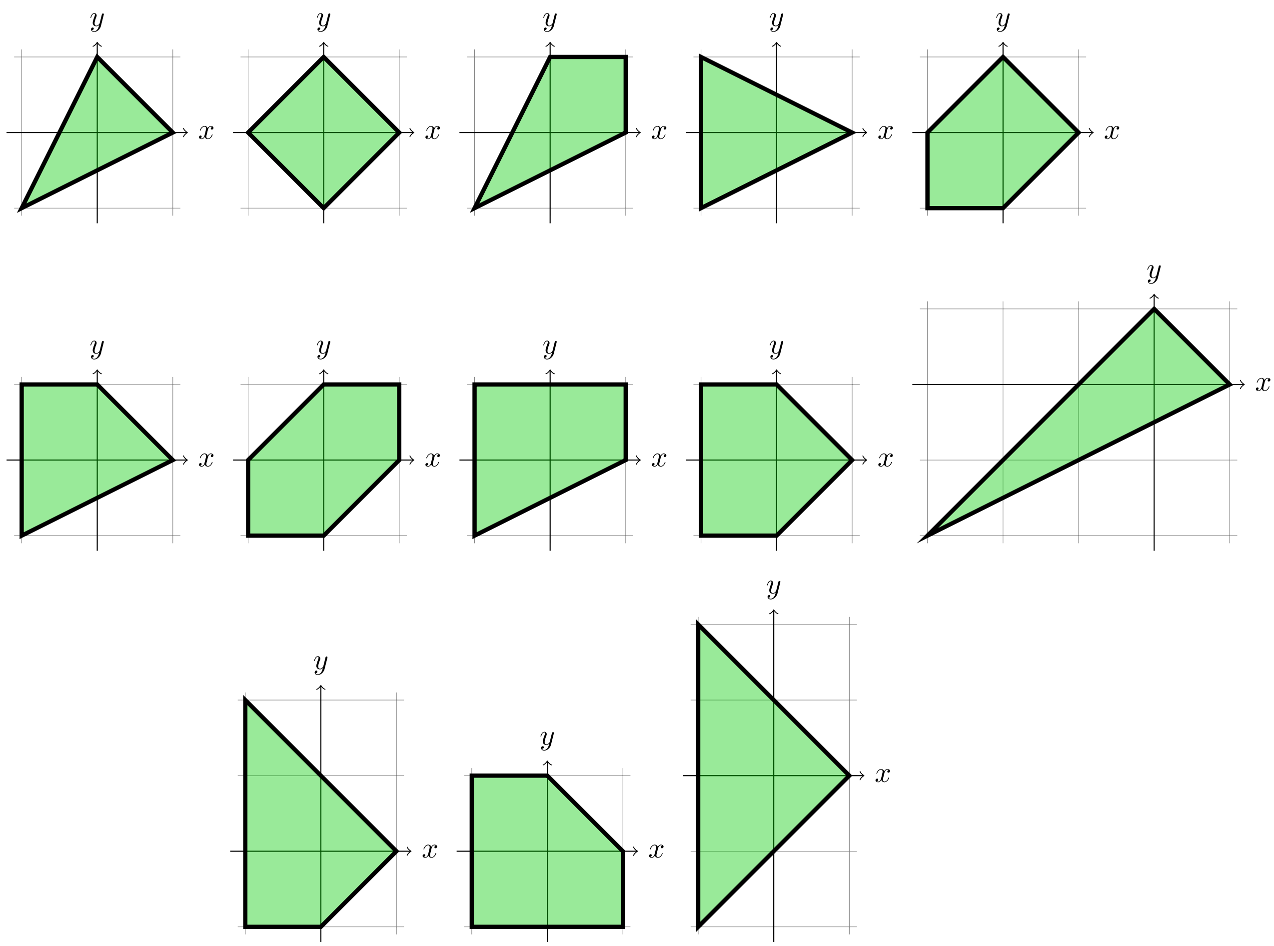
Definition 1.2.0.7 (Unimodular)**.** *Consider a set of unimodular matrices; matrices with integer entries and determinant equal to either 1 or -1. This set form a group* $GL_2(\mathbb{Z})$ *under multiplication.*

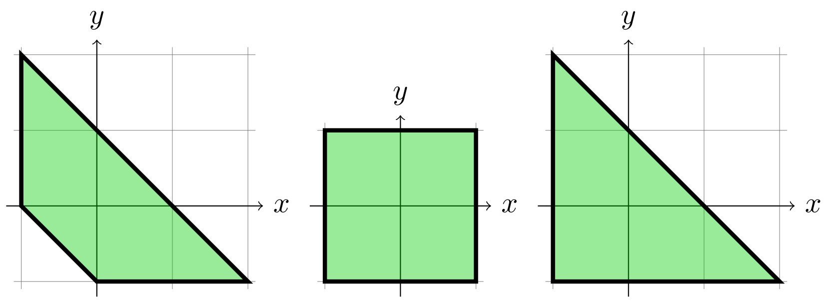
We consider any two *reflexive polygons* to be isomorphic to each other if there exists an unimodular map that sends one to another. It is intuitive to classify *reflexive polygons* in this way since the unimodular map preserves the natural characterstics of the original *reflexive polygon*, such as the number of boundary points and interior points(hence, the area as well by Pick's Theorem).

Proposition 1.2.0.1 (Classification)**.** *Up to equivalence, there are total of 16 reflexive polygones.*

It is fairly easy to prove this proposition by hand. Consider *reflexive polygons* with 3 boundary points(only one such reflexive polygon), consider for 4 boundary points, and so on up to 9 boundary points.

Below are the 16 *reflexive polygons*.





This is a remarkable feat; by classifying equivalence classes of *reflexive polygons* we now know each and every one of them and that there are only 16 in total!

Remark 1.2.0.1 (Higher Dimensions)**.** *Although this paper focuses on reflexive polytopes of dimension 2, symmetric behavior of the dual pair holds for higher dimensions; both the original reflexive polytope and its dual are lattice polytopes with the origin as its only interior lattice point (A little word of caution: for dimension $n \geq 3$, a lattice polytope with the origin as its only interior point does not necessarily imply its dual is a reflexive polytope as well. So the definition P is reflexive if and only if P^* is reflexive' is appropriate in this case). For any arbitrary dimension n, two reflexive polytopes are isomorphic to each other if there exists a matrix in $GL_n(\mathbb{Z})$ that maps one to another.*
So how many are there up to isomorphism? With the help of a computer[1]*, it is known that for $n = 3$ there are 4319 isomorphism classes, and 473800776 for $n = 4$. However, it is not known how many there are for $n \geq 5$.*

[1]For an algorithm for classifying reflexive polytopes, please look at [10]

1.3. The number 12

Suppose that we are given a *reflexive polygon* P whose boundary points are represented by the vectors $p_1, p_2, \cdots, p_B$ (in counter-clockwise order from the x-axis). Then, P^* is defined as the convex hull of $q_i = p_{i+1} - p_i$ for $1 \leq i \leq B$ and for $i = B$, $q_B = q_1 - q_B$ (Well not quite. Remember an easy way of finding the dual? We computed q_i's then rotated them 90 degrees clockwise to obtain the dual. But we know that this rotation is an unimodular map. So the convex hull of q_i's is isomorphic to the actual dual). Let A, I, and B denote the area of P, the number of interior points of P, and the number of boundary points of P, respectively. Define A^*, I^*, and B^* for P^* in the same manner. We now present a very well known property of a *reflexive polygon.*

Theorem 1.3.0.1 (Number 12)**.** *Let P be a reflexive polygon and P^* its dual. Then $B + B^* = 12$.*

There are many different proofs of this Theorem. Easiest one, of course, is an exhaustive proof: since there are only 16 *reflexive polygons* we need to check 16 times to confirm that the Theorem holds. Other proofs are given by the use of Noether's formula(special case of the Hirzebruch-Riemann-Roch) in toric varieties[2] or by Modular Forms[3].

In this paper, we will go over an elementary proof given in the paper [2] with a slight modification[4]. We will observe that as B increases by one B^* decreases by one, so that their sum remains at 12.

Theorem 1.3.0.2 (Pick's Theorem)**.** *Given a lattice polygon P, its area is given by $A = I + \frac{B}{2} - 1$.*

For *reflexive polygons* we simply get $A = \frac{B}{2}$. We realize that there's a different way to represent the area.

Proposition 1.3.0.2. *Given a lattice polygon P with boundary points $p_1, p_2, \cdots, p_B$, its*

[2] Look at [11] and [12] for the proof

[3] Proof is given in [4], a wonderful paper written by brilliant people

[4] It presents the proof in the complex plane, which didn't seem necessary(there are some typos as well)

area is also given by

$$A = \frac{1}{2}\sum_{i=1}^{B} A_{i,i+1}$$

where we denote $A_{i,i+1} = det \begin{pmatrix} x_i & y_i \\ x_{i+1} & y_{i+1} \end{pmatrix} = x_i y_{i+1} - x_{i+1} y_i$, *for* $p_i = (x_i, y_i)$.

Proof. Given vectors (a, b) and (c, d) we know that the parallelogram formed by these vectors has area $ad - bc$. So the area of the triangle formed by the origin, (a, b) and (c, d) is $\frac{1}{2}(ad - bc)$. Since P has boundary points $p_1, p_2, \cdots, p_B$ and its area is the sum of all triangles formed by the origin and two consecutive vectors:

$$A = \frac{1}{2}\sum_{i=1}^{B} x_i y_{i+1} - x_{i+1} y_i$$

. ☐

Similarly, using the fact that P^* has boundary points $q_i = p_{i+1} - p_i = (x_{i+1}, y_{i+1}) - (x_i, y_i) = (x_{i+1} - x_i, y_{i+1} - y_i)$, we obtain

$$\begin{aligned} A^*_{i,i+1} &= (x_{i+1} - x_i)(y_{i+2} - y_{i+1}) - (x_{i+2} - x_{i+1})(y_{i+1} - y_i) \\ &= x_{i+1}y_{i+2} - x_{i+1}y_{i+1} - x_i y_{i+2} + x_i y_{i+1} - x_{i+2}y_{i+1} + x_{i+2}y_i + x_{i+1}y_{i+1} - x_{i+1}y_i \\ &= A_{i,i+1} + A_{i+1,i+2} - A_{i,i+2} \end{aligned} \tag{1.1}$$

Hence, the area of the dual is

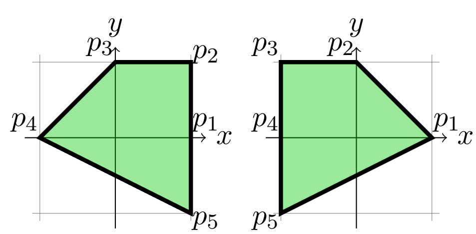
$$A^* = \frac{1}{2}\sum_{i=1}^{B} A^*_{i,i+1} = \frac{1}{2}\sum_{i=1}^{B}(A_{i,i+1} + A_{i+1,i+2} - A_{i,i+2}) = \sum_{i=1}^{B} A_{i,i+1} - \frac{1}{2}\sum_{i=1}^{B} A_{i,i+2} = 2A - \frac{1}{2}D$$

where we used the fact $\sum_{i=1}^{B} A_{i,i+1} = \sum_{i=1}^{B} A_{i+1,i+2}$ and denoted D as $\sum_{i=1}^{B} A_{i,i+2}$. Now we use the *Pick's Theorem* to represent B^* in terms of B:

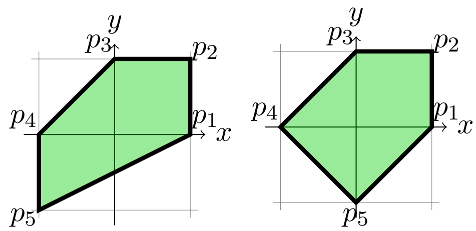
$$\frac{B^*}{2} = 2\frac{B}{2} - \frac{1}{2}D \implies B^* = 2B - D$$

As stated before, we hope to observe that as B increases by 1 B^* decreases by 1. This would be equivalent to observing that D increases by 3 as B increases by 1.

First, we consider the case when $B = 3$. Then, $D = A_{1,3} + A_{2,1} + A_{3,2} = -(A_{3,1} + A_{1,2} + A_{2,3}) = -2A = -2(\frac{B}{2}) = -B = -3$ (In fact, $A_{i,i+1} = 1$ for all i since $A = \frac{1}{2}\sum_{i=1}^{B} A_{i,i+1} \implies B = \sum_{i=1}^{B} A_{i,i+1}$ and each $A_{i,i+1}$ represents the triangle formed by the origin, point i, and point $i+1$, so it is positive. There are B many terms in the sum, so each $A_{i,i+1}$ must be 1). Thus, $B^* = 2 \cdot 3 - (-3) = 9$ and $B + B^* = 12$ as expected. When $B = 4$, $D = A_{1,3} + A_{2,4} + A_{3,1} + A_{4,2} = 0$ so $B^* = 2 \cdot 4 = 8$.
For the case $B = 5$, we consider all possible *reflexive polygons* with 5 boundary points and deduce that D must be 3. First, we realize that no edge can have more than 3 boundary points(if an edge had 4 boundary points, the last remaining point can't be placed so that P has 5 boundary points and contains the origin), and that we may center P at the origin with boundary points on the unit square(there's only 2 *reflexive polygons* with 5 boundary points and both are contained within the unit square). So we have 5 points to be plotted on 8 points of the unit square. This implies that at least one of the 4 lines $x = 0$, $y = 0$, $y = \pm x$ must contain 2 points. Assume that this is true for the x-axis. Then, two of the three leftover points must be either be above or belowe the x-axis. Assume that they are above the x-axis.
We look at the case when an edge has 3 vertices.



For the first *reflexive polygon* above, $D = A_{1,3}+A_{2,4}+A_{3,5}+A_{4,1}+A_{5,2} = 1+1-1+0+2 = 3$, and for the second, $D = 1 + 1 + 2 + 0 - 1 = 3$.
When P has no edge with 3 vertices:

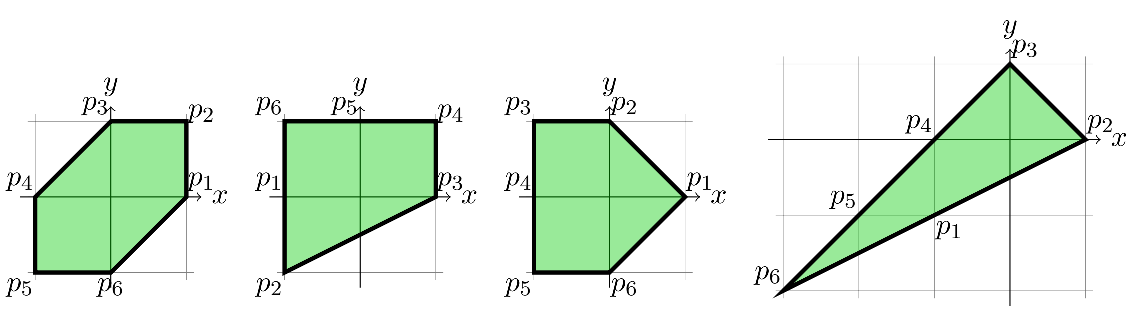


By the same analysis, the first *reflexive polygon* has $D = 1+1+1+0+0 = 3$, and the second has $D = 1+1+0+0+1 = 3$.

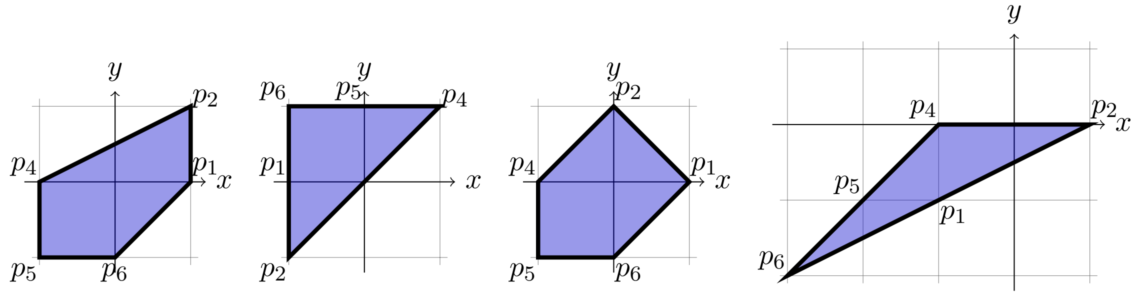
Hence, in an exhaustive fashion, we show that $D = 3$ when $B = 5$, which is what we wanted to show.

When $B = 6$, we label the boundary points(counter-clockwise) such that both p_6 and p_3 are vertices of P. Then, we delete p_6 from P to obtain a convex lattice polygon $P^{'}$ of 5 boundary points containing the origin, which is a convex hull of $\{p_1, p_2, p_3, p_4, p_5\}$. Then, $D - D^{'} = A_{1,3} + A_{2,4} + A_{3,5} + A_{4,6} + A_{5,1} + A_{6,2} - (A_{1,3} + A_{2,4} + A_{3,5} + A_{4,1} + A_{5,2}) = A_{4,6} + A_{5,1} + A_{6,2} + A_{1,4} + A_{2,5}$. We can consider the last sum as $D^{''}$ where $P^{''}$ is either the lattice pentagon of the convex hull of $\{p_1, p_2, p_4, p_5, p_6\}$ containing the origin, or the lattice polygon with 6 boundary points containing no interior points.

It is time to look at some examples to firmly understand what was stated in the above paragraph. Below are four unique *reflexive polygons* with 6 boundary points, where each are labeled so that p_3 and p_6 are vertices.



Then, we delete the point p_3 to obtain $P^{''}$, the convex hull of $\{p_1, p_2, p_4, p_5, p_6\}$.



When $P^{''}$ is a lattice pentagon, as in the first and the third cases, we know that it has $D^{''} = 3$, and when it is a lattice hexagon containing no interior point, as in case 2 and 4, $D^{''} = A_{4,6} + A_{5,1} + A_{6,2} + A_{1,4} + A_{2,5}$ is computed to be 3 as well.

Since $P^{'}$ is a *reflexive polygon* with 5 boundary points, $D^{'} = 3$. Therefore, we have $D = D^{'} + D^{''} = 3 + 3 = 6$ so that $B^* = 2 \cdot 6 - 6 = 6$.

When $B = 7, 8, 9$, we observe an identical behavior. First, we label the boundary points such that p_B is a vertex of P and at least one of $p_3, p_4, \cdots, p_{B-3}$ is a vertex of P as well. Then, create a *reflexive polygon* $P^{'}$ with $B - 1$ boundary points by deleting p_B from P. Next, we consider $P^{''}$, a convex hull of $\{p_1, p_2, p_{B-2}, p_{B-1}, p_B\}$ with corresponding $D^{''} = D - D^{'}$. This $P^{''}$ will always be either a *reflexive polygon* of 5 boundary points or a lattice hexagon with no interior points. In either case, $D^{''} = 3$, which implies that $D = D^{'} + 3$ showing that D increase by 3 as B increase by 1.

1.4. Hibi's Palindromic Theorem

In this section, we will generalize *Hibi's Palindromic Theorem* to a *reflexvie polytope* of any dimension d. But first we need some definitions and theorems to understand the background of integer-point enumeration[5].

Definition 1.4.0.8 (lattice-point enumerator)**.** *We denote the* ***lattice-point enumerator*** *for the t^{th} dilates of a polytope $P \subset R^d$ by $L_P(t) := \#(tP \cap \mathbb{Z}^d)$*

[5]There are some theorems whose proofs will not be presented. It would simply take too long to cover all the necessary backbones of integer-point enumeration, which is not the intention of this paper. Please look at Ch.2 and Ch.3 of [1] for detailed explanations

Definition 1.4.0.9 (Ehrhart)**.** *We consider the generating function of L_P:*

$$Ehr_P(z) := \sum_{t\geq 0} L_P(t)z^t$$

This generating function is also called the ***Ehrhart series*** *of P.*

Example 1.4.0.2. *Consider a unit d-cube $P = \square$. Then, $L_\square(t) = (t+1)^d$. It is a fun exercise to check that*

$$Ehr_\square(z) = \sum_{t\geq 0}(t+1)^d z^t = \frac{1}{z}\sum_{t\geq 1} t^d z^t = \frac{\sum_{k=1}^{d} A(d,k)z^{k-1}}{(1-z)^{d+1}}$$

where $A(d,k)$ is the ***Eulerian number****. This is because repeating the steps, differentiating $\sum_{t\geq 1} z^t = \frac{z}{1-z}$ and multiplying by z, d many times gives us $\sum_{t\geq 1} t^d z^t = \frac{\sum_{k=1}^{d} A(d,k)z^k}{(1-z)^{d+1}}$. Also, the number of interior lattice points is $L_{\square^\circ}(t) = (t-1)^d$, so that we get a reciprocal relation $L_\square(-t) = (-1)^d L_{\square^\circ}(t)$.*

Definition 1.4.0.10 (pointed cone)**.** *A* ***pointed cone*** *$K \subseteq \mathbb{R}^d$ is a set*

$$K = \{\boldsymbol{v} + \lambda_1\boldsymbol{w}_1 + \lambda_2\boldsymbol{w}_2 + \cdots + \lambda_m\boldsymbol{w}_m : \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0\}$$

where $\boldsymbol{v}, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m \in \mathbb{R}^d$. $\boldsymbol{v}$ is called an ***apex*** *of K and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m$ generators of K.*

A *Pointed cone* is very useful in enumerating lattice points of any d-polytope as we can perform a process called *coning over a polytope.*

Definition 1.4.0.11 (cone over)**.** *Let $P \subset \mathbb{R}^d$ be a convex polytope with vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$. We can lift theses vertices into $\mathbb{R}^{d+1}$ by letting $\boldsymbol{w}_1 = (\boldsymbol{v}_1, 1)$, $\boldsymbol{w}_2 = (\boldsymbol{v}_2, 1)$, $\ldots$, $\boldsymbol{w}_n = (\boldsymbol{v}_n, 1)$. We define the process* ***cone over*** *P by*

$$cone(P) = \{\lambda_1\boldsymbol{w}_1 + \lambda_2\boldsymbol{w}_2 + \cdots + \lambda_n\boldsymbol{w}_n : \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0\}$$

Hence, we can create a *pointed cone* $cone(P) \subset \mathbb{R}^{d+1}$ with $\mathbf{0}$ as its apex for any polytope $P \subset \mathbb{R}^d$. Since we have lifted the vertices of P to one dimensional higher by adding a 1 as their last coordinate, we can recover our polytope P from $cone(P)$ by slicing it with a hyperplane $x_{d+1} = 1$. Similarly, with a hyperplane $x_{d+1} = 2$ we can obtain $2P$, a dilate of P by 2. In general, with a hyperplane $x_{d+1} = t$ we can obtain tP (a way to understand why this is true: by letting $\lambda_i = t$ and all the other λ's zero for each $1 \leq i \leq n$, we can obtain vertices of tP. In other words, restricting the sum of λ_i's to t, which will cause the last coordinate of $\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_m \mathbf{w}_n$ to be t, *cone*(P) is precisely the cone over P sliced by $x_{d+1} = t$, or the dilate of P by t).

Now we want to utilize these dilates of P that we can generate from *cone*(P) to form a relational property between *cone*(P) and $Ehr_P(z)$. To do so we look at the following multivariate generating function.

Definition 1.4.0.12 (integer-point transform)**.** *The* ***integer-point transform*** *of P lists all integer points in P and is given by*

$$\sigma_P(\boldsymbol{z}) = \sigma_P(z_1, z_2, \ldots, z_d) := \sum_{\boldsymbol{m} \in P \cap \mathbb{Z}^d} \boldsymbol{z}^{\boldsymbol{m}}$$

Consider the *integer-point transform* of *cone*(P); $\sigma_{cone(P)}(z_1, z_2, \ldots, z_{d+1})$. Since the last coordinate of $\mathbf{m} \in cone(P) \cap \mathbb{Z}^d$ only takes on the values $0, 1, 2, \ldots$, that is, the integer points of *cone*(P) are only contained in the dilate copies of P, we can rewrite the *integer-point transform* of *cone*(P) in the following way:

$$\sigma_{cone(P)}(z_1, z_2, \ldots, z_{d+1}) = \sum_{\mathbf{m} \in cone(P) \cap \mathbb{Z}^{d+1}} (z_1^{m_1}, z_2^{m_2}, \ldots, z_{d+1}^{m_{d+1}}) = \sum_{t \geq 0} \sigma_{tP}(z_1, z_2, \ldots, z_d) z_{d+1}^t$$

We realize that plugging in 1 for all z_i's in $\sigma_P(z_1, z_2, \ldots, z_d)$ will give us the number of integer points in P. Thus, we have

$$\sigma_{cone(P)}(1, 1, \ldots, 1, z_{d+1}) = \sum_{t \geq 0} \sigma_{tP}(1, 1, \ldots, 1) z_{d+1}^t = \sum_{t \geq 0} \#(tP \cap \mathbb{Z}^d) z_{d+1}^t = \sum_{t \geq 0} L_P(t) z_{d+1}^t$$

In other words, $\sigma_{cone(P)}(1, 1, \ldots, 1, z) = Ehr_P(z)$.[6]

[6] Using this property, we can prove the *Ehrhart's Theorem* which states that $L_P(t)$ is a polynomial in t of degree d if P is a convex d-polytope with integer vertices. For this reason $L_P(t)$ is usually called the *Ehrhart polynomial.* The proof is given in section 3.3 in [1]

Theorem 1.4.0.3 (Stanley reciprocity)**.** *Suppose K is a rational d-cone with the origin as apex. Then*

$$\sigma_K(\frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_d}) = (-1)^d \sigma_{K^\circ}(z_1, z_2, \ldots, z_d)$$

where $\sigma_{K^\circ}(z_1, z_2, \ldots, z_d)$ represents the integer-point transform of the interior of K.

We will take this theorem for granted.[7]

Theorem 1.4.0.4. *Let P be a convex rational d-polytope. Then,*

$$Ehr_P(\frac{1}{z}) = (-1)^{d+1} Ehr_{P^\circ}(z)$$

where the Ehrhart series *for the interior of P is given as $Ehr_{P^\circ}(z) := \sum_{t\geq 1} L_{P^\circ}(t) z^t$.*

Proof. By the use of $\sigma_{cone(P)}(1, 1, \ldots, 1, z) = Ehr_P(z)$ and Stanley reciprocity, we have

$$Ehr_P(\frac{1}{z}) = \sigma_{cone(P)}(1, 1, \ldots, 1, \frac{1}{z}) = (-1)^{d+1} \sigma_{(cone(P))^\circ}(1, 1, \ldots, 1, z) = (-1)^{d+1} Ehr_{P^\circ}(z)$$

□

Finally, now we have all the basic tools to state and prove the *Hibi's palindromic theorem.*

Theorem 1.4.0.5 (Hibi's palindromic)**.** *Let P be a convex d-polytope with integer vertices containing the origin in its interior. Stanley showed in [3] that a polytope with integer vertices has the Ehrhart series*[8]

$$Ehr_P(z) = \frac{h_d z^d + h_{d-1} z^{d-1} + \cdots + h_1 z + h_0}{(1-z)^{d+1}}$$

Then P is a reflexive polytope if and only if $h_k = h_{d-k}$ for all $0 \leq k \leq \frac{d}{2}$.

Proof. First, we need to describe the *reflexive polytopes* in a more concrete way. We call a polytope P reflexive if it has the half-space description $P = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} \leq \mathbf{1}\}$ where $\mathbf{A}$ is an integer matrix and $\mathbf{1}$ denotes a vector with coordinates all equal to 1.

[7] The proof of *Stanley reciprocity* is given in [1] pg.86. Furthermore, there's a stronger statement known as *Ehrhart-Macdonald reciprocity*: $L_P(-t) = (-1)^{dimP} L_{P^\circ}(t)$

[8] It is known that given such P with this *Ehrhart series*, $h_0, h_1, \ldots, h_d \geq 0$, $h_1 = L_P(1) - d - 1$, $h_0 = 1$, and $vol(P) = \frac{1}{d!}(h_d + h_{d-1} + \cdots + h_1 + 1)$

For example, the *reflexive polygon* from Example 1.1.0.1 can be described by three inequalities: $y - x \leq 1$, $y + 2x \leq 1$, and $-2y - x \leq 1$. So, $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ -2 & -1 \end{pmatrix}$

In other words, each row of A is given by the integer coefficients a_i's in each facet hyperplane H of P where $H = \{\mathbf{x} \in \mathbb{R}^d : a_1x_1 + a_2x_2 + \cdots + a_dx_d = 1\}$.

We claim that P has this half-space descripition if and only if

$$P^\circ \cap \mathbb{Z}^d = \{\mathbf{0}\} \text{ and for all } t \in \mathbb{Z} > 0, (t+1)P^\circ \cap \mathbb{Z}^d = tP \cap \mathbb{Z}^d$$

Suppose to the contrary that P does not strictly contained the origin in its interior. Then, there is some $\mathbf{x} = (x_1, x_2, \ldots, x_d) \neq \mathbf{0}$ belonging in $P^\circ \cap \mathbb{Z}^d$ such that $\mathbf{Ax} \leq \mathbf{1}$ where $\mathbf{A}$ is some k x d integer matrix(for integer $k \geq d+1$). In order to contain this $\mathbf{x}$ within the interior of P there must be a facet hyperplane H that lies 'outside' of $\mathbf{x}$. That is, there is some integer point $\mathbf{y} \in H$ so that the absolute value of some coordinate y_j in $\mathbf{y}$ is greater than the absolute value of x_j in $\mathbf{x}$. But this is a contradiction since the condition $\mathbf{Ay} \leq \mathbf{1}$ cannot be met. Hence, $P^\circ \cap \mathbb{Z}^d = \{\mathbf{0}\}$ must hold.

We give a precise definition of the interior of P as $P^\circ = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{Ax} < \mathbf{1}\}$. So the second condition is equivalent to $\{\mathbf{x} \in \mathbb{Z}^d : \mathbf{Ax} < \mathbf{t+1}\} = \{\mathbf{x} \in \mathbb{Z}^d : \mathbf{Ax} \leq \mathbf{t}\}$. Since $\mathbf{A}$ is an integer matrix and $\mathbf{x} \in \mathbb{Z}^d$, the resulting vector of $\mathbf{Ax}$ has all integer coordinates. Hence, if the coordinates of $\mathbf{Ax}$ are less than $t+1$ they must be less than or equal to t.

Now proving the other way: assume that $P^\circ \cap \mathbb{Z}^d = \{\mathbf{0}\}$ and $(t+1)P^\circ \cap \mathbb{Z}^d = tP \cap \mathbb{Z}^d$ for all $t \in \mathbb{Z} > 0$, where P is an integer d-polytope. This mean that the only lattice points that we gain are the boundary points of $(t+1)P$ as we transit from tP to $(t+1)P$. So, there are no lattice points between tH and $(t+1)H$ for any facet hyperplane H. In other words, given a facet hyperplane $H = \{\mathbf{x} \in \mathbb{R}^d : a_1x_1 + a_2x_2 + \cdots + a_dx_d = b\}$[9], where we may assume $gcd(a_1, a_2, \ldots, a_d, b) = 1$(if gcd is not 1, we can cancel out that factor), we have

$$\{\mathbf{x} \in \mathbb{Z}^d : tb < a_1x_2 + a_2x_2 + \cdots + a_dx_d < (t+1)b\} = \varnothing$$

[9]Here, the values $a_1, a_2, \ldots, a_d, b$ are taken to be integers in order for the intersections of the facet hyperplanes to produce integer vertices of P. If these values are rational we can simply make them into integers, but certainly they can't be irrational

The only possible value of b that fits the condition above is $b = 1$. Therefore, P must have the half-space description $P = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} \leq \mathbf{1}\}$.

Now to prove Hibi's Thoerem: by Theorem 1.4.0.4 and definition $(t+1)P^\circ \cap \mathbb{Z}^d = tP \cap \mathbb{Z}^d$,

$$
\begin{aligned}
Ehr_{P^\circ}(z) = \sum_{t\geq 1} L_{P^\circ}(t)z^t = (-1)^{d+1} Ehr_P(\frac{1}{z}) = \frac{h_0 z^{d+1} + h_1 z^d + \cdots + h_{d-1}z^2 + h_d z}{(1-z)^{d+1}} \\
= \sum_{t\geq 1} L_P(t-1)z^t = z\sum_{t\geq 0} L_P(t)z^t = \frac{h_d z^{d+1} + h_{d-1} z^d + \cdots + h_1 z^2 + h_0 z}{(1-z)^{d+1}}
\end{aligned}
\tag{1.2}
$$

Thus, we see that P is reflexive if and only if $h_k = h_{d-k}$ for all $0 \leq k \leq \frac{d}{2}$. □

Remark 1.4.0.2. Reflexive polytopes *drew many attentions from physicists due to their applications to mirror symmetry in string theory. They have a deep connection with toric geometry in that the toric variety defined by a* reflexive polytope *is* Fano*, and that every generic hypersurface of this toric variety is* Calabi-Yau*. For more info, please look at [13]. It was shown that the sum of the numbers of boundary points of P and P^* is always 12 in dimension 2. Similarly, in dimension 3, $\sum_e length(e) \cdot length(e^*) = 24$ where e and e^* are the edge of P and P^*, respectively. No similar result is known for dimension greater than 3.*

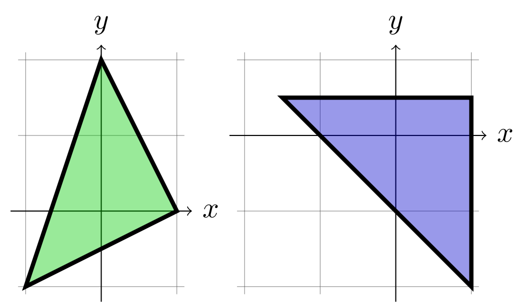
2. L-REFLEXIVE POLYGONS AND LOOPS

L-reflexive Polygons And Loops

2.1. l-reflexive polygons

Now that we are familiar with the notion of *reflexive polytopes*, it is natural to wonder if we can extend the properties of *reflexive polytopes* to a more general setting. For example, can we find a similar result for the lattice polygons with two interior lattice points?

Example 2.1.0.3. *Consider a lattice polygon with vertices* $(1,0)$*,* $(0,2)$*, and* $(-1,-1)$*. Its dual, by definition, is the intersection of half-spaces* $x \leq 1$*,* $2y \leq 1$*, and* $-x-y \leq 1$*.*



As we see, the dual is not a lattice polygon and the number of boundary points do not add up to 12. We realize that the point $(0,2)$ *has given us a half-space* $2y \leq 1$*, which caused the dual to have non-integer vertices.*

From now on, let N be a lattice and M its dual lattice. We define a lattice point x in $N \setminus \{\mathbf{0}\}$ to be **primitive** if the line segment from $\mathbf{0}$ to x contains no other lattice points. In other words, x is primitive if gcd of its coordinates is 1. We define a **primitive outer normal** of a facet F of P to be the unique primitive lattice point $u_F \in M$ such that $F = \{x \in P : \langle u_F, x \rangle = l_F\}$ for some unique $l_F \in \mathbb{Z} > 0$. u_F is normal in a sense that it is a vector that is normal to the facet hyperplane of F. l_F is called the **local index** of F; it is the integral distance of $\mathbf{0}$ from the facet hyperplane. The **index** of P is defined as the least common multiple of the l_F's.

In the above example, to find u_F for each facet F, we first need to find what the facet hyperplanes are. The hyperplanes are $y = -2x + 2$, $y = 3x + 2$, and $\frac{1}{2}x - \frac{1}{2}$. So the corresponding normal lines passing through the origin are $y = \frac{1}{2}x$, $y = -\frac{1}{3}x$, and $y = -2x$, and the u_F's are $(2, 1)$, $(-3, 1)$, and $(1, -2)$. Then, l_F's are found to be 2,2, and 1 so that this lattice polygon has index 2. Of course, no index can be defined for its dual since the dual is not a lattice polygon.

Definition 2.1.0.13 (l-reflexive polytope)**.** *A lattice polytope is called l-**reflexive**[10] or a reflexive polytope of index l if, for some $l \in \mathbb{Z} > 0$, the following conditions hold:*

(a) P contains the origin in its interior

(b) The vertices of P are primitive

(c) For any facet F of P the local index l_F equals l

We notice that our existing notion of *reflexive polytope* is equivalent to *1-reflexive polytope*; the conditions (a),(b),(c) are met by the definition of *reflexive polytope* $P = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{Ax} \leq \mathbf{1}\}$.

Proposition 2.1.0.3 (duality)**.** *Let P be a lattice polytope with primitive vertices and contain the origin in its interior. Then P is l-reflexive if and only if lP^* is a lattice polytope with primitive vertices. In this case, lP^* is also l-reflexive. That is, P and lP^* form a natural duality between them.*

Proof. We note that for any polytope P of degree d the vertices of P^* are the points $\frac{u_F}{l_F}$: let $(x_{11}, x_{12}, \ldots, x_{1d}), (x_{21}, x_{22}, \ldots, x_{2d}), \ldots, (x_{d1}, x_{d2}, \ldots, x_{dd})$ be the vertices of some facet F of P (there are at least d many vertices in a facet of a d-polytope). Also let $u_F = (a_1, a_2, \ldots, a_d)$. For some $y = (p_1, p_2, \ldots, p_d)$ in the dual lattice M, we have that $p_1x_{11} + p_2x_{12} + \ldots + p_dx_{1d} \leq 1, p_1x_{21} + p_2x_{22} + \ldots + p_dx_{2d} \leq 1, \ldots, p_1x_{d1} + p_2x_{d2} + \ldots + p_dx_{dd} \leq 1$ by the definition of P^*. So, we get a vertex of P^* out by the intersection of the hyperplanes $p_1x_{11} + p_2x_{12} + \ldots + p_dx_{1d} = 1, p_1x_{21} + p_2x_{22} + \ldots + p_dx_{2d} = 1, \ldots, p_1x_{d1} + p_2x_{d2} + \ldots + p_dx_{dd} = 1$. By the definition of F we have that u_F satisfy $a_1x_{11} + a_2x_{12} + \ldots + a_dx_{1d} = l_F, a_1x_{21} + a_2x_{22} + \ldots + a_dx_{2d} = l_F, \ldots, a_1x_{d1} + a_2x_{d2} + \ldots + a_dx_{dd} = l_F$. This shows that $(\frac{a_1}{l_F}, \frac{a_2}{l_F}, \ldots, \frac{a_d}{l_F}) = \frac{u_F}{l_F}$ is the vertex of the intersection of the hyperplanes described above. This shows that we can think of the facets of P^* as being in one-to-one

[10] The existence of 1-reflexive have been known for some time, but the exsitence of l-reflexive was first found by Alexander Kasprzyk and Benjamin Nill in [5]

correspondence with the vertices of P.

Suppose P is l-reflexive, so that all $l_F = l$. Then lP^* is a convex hull of all u_F's. Hence, the vertices of lP^* are primitive. By the one-to-one correspondence, we have that a facet of lP^* is described as $\{x \in lP^* : \langle v, x\rangle = l\}$ where v is a vertex of P. Since any vertex of P is primitive, we have that each facet of lP^* has local index l. Thus, lP^* is l-reflexive.

Conversely, suppose lP^* is l-reflexive. Analogous to the argument above, for any facet F of P we have that $l(\frac{u_F}{l_F})$ is the unique primitive lattice point. Thus, $l_F = l$ and P is l-reflexive.

Finally, a natural duality follows since $l(lP^*)^* = l(P)^* = l(\frac{1}{l}P) = P$. (we also showed how to find the corresponding lP^* of *l-reflexive polygon* P in a really easy way. lP^* is simply a convex hull of the u_F's) □

This is a remarkable achievement as now we are now able to characterize some lattice polytopes which contains more than one interior points. Proposition 2.1.0.3 is very satisfying in a sense that the magical number of 12 holds in a more general setting of *reflexive polytopes* of higher index. The proof of this fact will be shown in a much more general setting later on.

Now that we know the existence of *l-reflexive polygons*, just as we found that there are 16 *1-reflexive polygons* up to isomorphism, can we classify *l-reflexive polygons* for any arbitrary l so that there are finitely many of them?

It has been proven in [14] that the finiteness of the number of equivalence classes holds for any lattice d-polytope of bounded volume. The basic idea of the proof is that any lattice polytopes of volume $\leq V$ is isomorphic to a lattice polytope contained in a lattice cube of side length at most $n \cdot n!V$ under a unimodular map(ignoring the position of the polytopes. That is, the map is $U\mathbf{x} + \mathbf{y}$ for $\mathbf{x} \in P$ and some $\mathbf{y} \in \mathbb{Z}^d$). Here, a lattice cube means that its sides are parallel to the coordinate axes. Since the number of lattice points of a lattice cube is finite, there are finitely many equivalence classes of such polytopes. This result implies that there are only finitely many l-reflexives of dimension d up to isomorphism.

Nonetheless, finding all the equivalence classes of *l-reflexive polygons* for each index l is challenging. Below is the table[11] for the number of isomorphism classes of *l-reflexive*

[11]taken from [5]

polygons for $1 \leq l < 60$.[12]

l	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29
$n(l)$	16	1	12	29	1	61	81	1	113	131	2	163	50	2	215
l	31	33	35	37	39	41	43	45	47	49	51	53	55	57	59
$n(l)$	233	2	34	285	3	317	335	2	367	182	3	419	72	4	469

We notice that there are no *l-reflexive polygons* of even index. We will look at two proofs of this phenomenon; one is elementary but a bit messy, and the other is very simple and uses a change of lattice.

The other interesting observation of the table above is the seemingly very slow growth of the number of 3k-reflexive polygons as k increases.

Theorem 2.1.0.6 (no even)**.** *There are no reflexive polygons of even index.*

Proof. Although the ultimate goal of the proof is to prove the theorem, the elementary proof itself will show many interesting properties of the *l-reflexive polygons.*

Let P be l-reflexive, then lP^* is l-reflexive. Let F be a facet of P with (x_1, y_1) and (x_2, y_2) as its vertices(going counter-clockwise). We want to firmly prove that u_F's are vertices of lP^*. By definition we have

$$F = \{x \in P : \langle u_F, x\rangle = l\} \text{ and } lP^* = \{y \in \mathbb{R}^2 : \langle y, x\rangle \leq l, x \in P\}$$

So we have that $ax_1 + by_1 = l$ and $ax_2 + by_2 = l$ by the definition of F. Recall that we find lP^* by computing the overlapping section of the half-spaces $\langle y, x\rangle \leq l$ where we take x to be a vertex of P. In other words, the vertices of lP^* are precisely the intersections of the facet hyperplane $\langle y, x\rangle = l$ for each x a vertex of P. Thus, the intersection of two facet hyperplanes created by (x_1, y_1) and (x_2, y_2) is precisely $u_F = (a, b)$. In other words, u_F is a vertex of lP^*.

Since we have two linear equations $ax_1 + by_1 = l$ and $ax_2 + by_2 = l$ we can solve for a and b. We get

$$a = \frac{-l(y_1 - y_2)}{x_1y_2 - x_2y_1} \text{ and } b = \frac{l(x_1 - x_2)}{x_1y_2 - x_2y_1}$$

[12] A complete classification up to index 200 is avaliable at http://www.grdb.co.uk/forms/toriclr2 which is a big database of graded rings in algebraic geometry

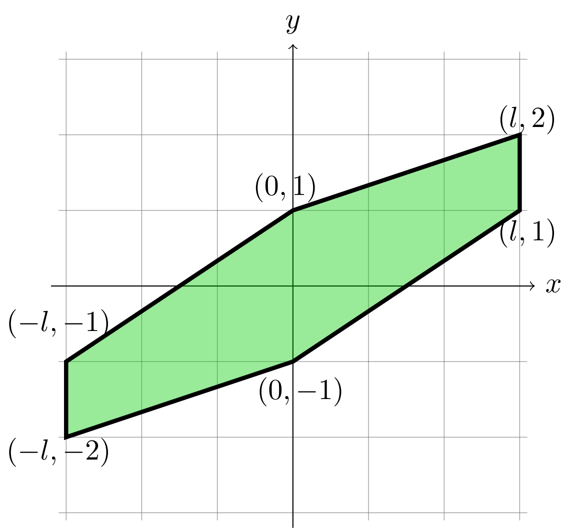
Suppose for now that (x_1, y_1) and (x_2, y_2) are subsequent lattice points on F(counter-clockwise). The equations for a and b are still satisfied for these lattice points. We want to show that $det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = x_1y_2 - x_2y_1$ is precisely l. Since a and b are integers we must have that $x_1y_2 - x_2y_1$ divide $-l(y_1 - y_2)$ and $l(x_1 - x_2)$. If $x_1y_2 - x_2y_1$ divides l we would have that a and b are not primitive, which is a contradiction. Hence, $x_1y_2 - x_2y_1$ must divide both $(y_1 - y_2)$ and $(x_1 - x_2)$. If we can show that it doesn't, then we must have that $x_1y_2 - x_2y_1 = l$. Showing that such division is not possible is equivalent to showing that $gcd(x_1 - x_2, y_1 - y_2) = 1$. But $gcd(x_1 - x_2, y_1 - y_2) + 1$ is precisely the number of points between(including) (x_1, y_1) and (x_2, y_2)[13], and these points are subsequent to one another so $gcd(x_1 - x_2, y_1 - y_2) + 1 = 2$. Therefore, for any two subsequent lattice points on P the determinant of them is l. So we can easily find the vertex $u_F = (a, b)$ of lP^* corresponding to the facet F of P by $a = y_2 - y_1$ and $b = x_1 - x_2$ for two subsequent points (x_1, y_1) and (x_2, y_2) on F. This explains the method shown in Example 1.1.0.1.
Now let (x_1, y_1) and (x_2, y_2) be vertices of F again. We know that by definition (x_1, y_1), (x_2, y_2), and (a, b) are primitive. Suppose that l is even. We want to get a contradiction out. The condition for a point to be primitive is that either both coordinates are odd or one coordinate odd and the other even. Suppose a is odd and b is even(or a even and b). Then, by the equation of a we must have that $y_1 - y_2$ is odd and $x_1y_2 - x_2y_1$ is even. $y_1 - y_2$ is odd if and only if one is odd and the other is even. Suppose y_1 is odd an y_2 is even(or y_2 is odd and y_1 is even). Then we must have that x_2 is odd since (x_2, y_2) is primitive. But then, $x_1y_2 - x_2y_1$ ends up being odd, which is a contradiction. Now suppose that both a and b are odd. By the same argument as above, $y_1 - y_2$ and $x_1 - x_2$ are odd. Again, assume that y_1 is odd and y_2 is even. Then, x_2 is odd so x_1 is even. Then, $x_1y_2 - x_2y_1$ ends up being odd again.
Therefore, l can't be even, meaning that theres is no *reflexive polygons* of even index. □

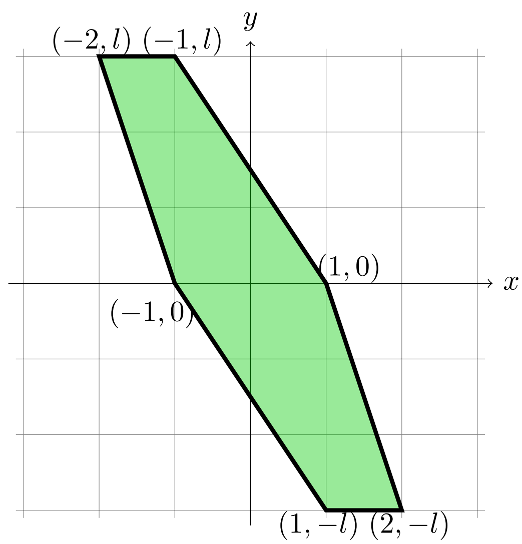
There are particularly interesting *l-reflexive polygons*, one for each odd index. Let's look at them.

Example 2.1.0.4 (centrally-symmetric hexagon). *Let P_l be a lattice polygon that is a*

[13] Translate these points to get $(0, 0)$ and $(x_1 - x_2, y_1 - y_2)$

convex hull of $\{\pm(0,1), \pm(l,2), \pm(l,1)\}$ *where* l *is odd.* P_l *is* l*-reflexive. No edge of* P_l *contains any lattice point, so we can quickly compute* lP_l^* *to be a convex hull of* $\{\pm(1,0), \pm(2,-1), \pm(1,-l)\}$.





We see that P_l *is centrally-smmetric hexagon, and is self-dual since it is isomorphic to* lP_l^**. Also, as shown in the table, this is the only possible* l*-reflexive for* $l = 3, 9, 15$*. Note that a unique 1-reflexive centrally-symmetric hexagon* $P := conv\{\pm(0,1), \pm(1,1), \pm(1,0)\}$ *can be mapped(morphed) into* P_l *by right multiplication with the matrix* $\begin{pmatrix} l & 1 \\ 0 & 1 \end{pmatrix}$*. This is an exciting observation, and we may suspect that there's a relationship between any* l*-reflexive and 1-reflexive.*

2.2. l-reflexive loops

We now present a generalization of *l-reflexive polygons* in a non-convex setting.

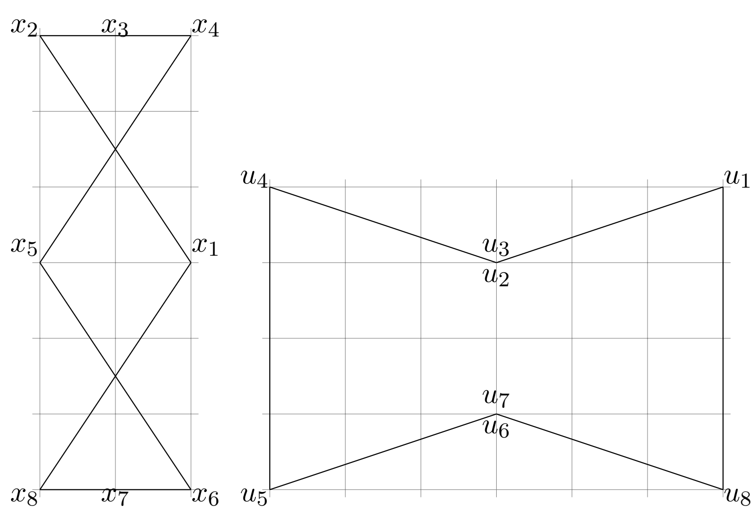
Definition 2.2.0.14. *Let* $\{x_1, \ldots, x_t\} \in N = \mathbb{Z}^2$ *be a boundary lattice points of* P*. We call* P *an* l***-reflexive loop of length t*** *if the following conditions are satisfied for* $1 \leq i \leq t$*:*

(i) The lattice point $x_{i+1} - x_i$ *is primitive*

(ii) *The determinant of the 2 x 2 matrix* A_i *formed by* x_i, x_{i+1} *equals* $\pm l$

(iii) *If* x_i *is a vertex then it is primitive.*

Here, the length t is defined as $\sum_{i=1}^{t} det(A_i)/l$. An *l-reflexive loop* may be non-convex or self-intersecting loop. Because of this we note that the definition of the dual is not well-suited since $lP^* = \{y \in \mathbb{R}^2 : \langle y, x\rangle \leq 1, x \in P\}$ is a convex polygon. Instead, we recall that the primitive outer normal u_F were precisely the vertices of lP^*. We define $\cup_{i=1}^{t} conv\{u_i, u_{i+1}\} \cap M$ for $i = 1, \ldots, t$ to be the set of boundary lattice points of lP^*. By the definition of 'primitive outer normal', given two boundary points (x_1, y_1) and (x_2, y_2), we find u_i by: considering the equation $y = \frac{x_1 - x_2}{y_2 - y_1} x$ since it is normal to the facet hyperplane of the two points, $(y_2 - y_1, x_1 - x_2)$ is a solution to the equation and to make it primitive we need to consider $\frac{1}{gcd(y_2 - y_1, x_1 - x_2)}(y_2 - y_1, x_1 - x_2)$ but the gcd is 1 by the definition (i) so $(y_2 - y_1, x_1 - x_2)$ is primitive normal, to ensure that it is outer we multiply by $\frac{l}{det(A_1)}$ which is either 1 or -1 depending on the direction of $x_1 x_2$. Thus, $\frac{l}{det(A_i)}(y_{i+1} - y_i, x_i - x_{i+1})$ for $1 \leq i \leq t$ are the boundary lattice points of lP^*. Below is an example of *l-reflexive loop* of length 0 and its corresponding dual of length 12.



The natural duality holds for the *l-reflexive loops* just as it did for the *l-reflexive polygons*. Now we dive into the world of lattice N and M that P and lP^* live in, to understand how *l-reflexive* is related to *1-reflexive*. From now on, let P be an *l-reflexive loop*.

Definition 2.2.0.15. *Let* Λ_P *be a lattice generated by the boundary points of* P.

Theorem 2.2.0.7. *$\Lambda_{lP^*} = l\Lambda_P^*$. Moreover, $\Lambda_P \subseteq N$ and $l\Lambda_P^* \subseteq M$ are both lattices of index l. The dual of Λ_P is defined as $\Lambda_P^* = \{y \in \mathbb{R}^2 : \langle x, y\rangle \in \mathbb{Z}, x \in \Lambda_P\}$.*

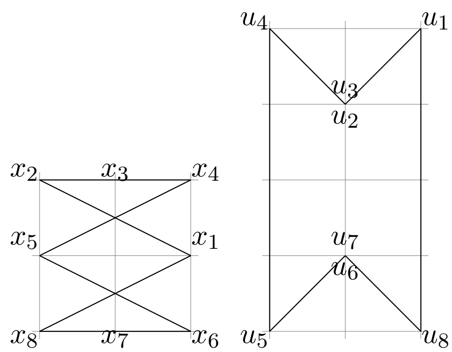
Proof. We want to show that $\Lambda_{lP^*} \subseteq l\Lambda_P^*$ and $l\Lambda_P^* \subseteq \Lambda_{lP^*}$.
First, we show that if $x \in \partial P \cap N$ (∂P denotes the boundary of P) and $u_F \in V(lP^*)$ ($V(lP^*)$ denotes the vertices of lP^*) then $\langle u_F, x\rangle \in l\mathbb{Z}$. We prove this inductively by showing that given a sequence of facets $F_1, \dots, F_s$ such that F_i is adjacent to F_{i-1}, $\langle u_{F_1}, x\rangle \in l\mathbb{Z}$ for $x \in F_{s+1} \cap N$ where F_{s+1} is the other facet adjacent to F_s.
Since u_{F_1} is primitive and a unimodular map is an integer matrix, there's a unimodular map that transforms u_{F+1} to be $(0, 1)$. So assume that $u_{F+1} = (0, 1)$. Let $u_{F_{S+1}} = (a, b), x = (c, d) \in F_{s+1} \cap N$, and $v = (k, l) \in V(F_1 \cap F_2)$. By applying the induction hypothesis to $F_{s+1}, F_s, \dots, F_2$, we get that l divides $\langle u_{F_{s+1}}, v\rangle = ak + bl$. So, l divides ak, and v is primtive so $gcd(k, l) = 1$, thus l divides a. Similarly, $\langle u_{F_{s+1}}, x\rangle = ac + bd = l$ means that l divides bd, and since $u_{F_{s+1}}$ is primtive we have that l divides $d = \langle u_{F_1}, x\rangle$. Thus, $\langle u_F, x\rangle \in l\mathbb{Z}$ for $x \in \partial P \cap N$ and $u_F \in V(lP^*)$.
Now let $x \in int(F) \cap N$ and $y \in int(G) \cap M$ (int means interior) for F a facet of P and G a facet of lP^*. We have $u_F \in V(lP^*)$ and $v_G \in V(P)$ as the corresponding primitive outer normals. Let $u_F = (a, b), v_G = (0, 1), x = (c, d), y = (k, l)$. We have that l divides $\langle v_G, u_F\rangle = b$ so that by $\langle u_F, x\rangle = ac + bd$, l divides ac. Since u_F is primitive and l divides b, we have $gcd(a, l) = 1$ which implies that l divides c. Therefore, l divides $\langle x, y\rangle = ck + dl$. This shows $\Lambda_{lP^*} \subseteq l\Lambda_P^*$.
To show the other direction, consider Λ_F, the lattice generated by the lattice points of F. Again, assume that $u_F = (0, 1)$ so that $(1, 0), (0, l)$ form a basis for Λ_F. Since l divides $\langle u_F, x\rangle$ for $x \in \partial P \cap N$, we have that $x \in \Lambda_P$. Hence, $\Lambda_P = \Lambda_F$. So, $\Lambda_P \subseteq N$ has index l and $\Lambda_{lP^*} \subseteq M$ has index l by mirror symmetry. Also we have that $(1, 0), (0, 1/l)$ form a basis for the dual Λ_P^*. Hence, we have that $l\Lambda_P^* \subseteq M$ is a lattice of index l as well. But then both Λ_{lP^*} and $l\Lambda_P^*$ having index l as sublattices of M means that they're in fact equal. □

The following corollaries follow as a consequence of Theorem 2.2.0.7.

Corollary 2.2.0.1. *A l-reflexive loop P is a 1-reflexive loop, say Q, with respect to the lattice Λ_P, which we call the 1-reflexive loop associated to P. Morover, Q^* a dual 1-*

reflexive loop of Q is isomorphic to the 1-reflexive loop associated to lP^*.

Proof. Let P be a l-reflexive loop. Assume that $u_F = (0,1)$ so that Λ_P is generated by $(1,0),(0,l)$. In this lattice, we can imagine l as being equivalent to 1 in $\mathbb{Z}^2$. Then, all three conditions of being 1-reflexive loop are satisfied. Hence, P is a 1-reflexive loop, say Q, with respect to Λ_P. Below is an depiction of P in Λ_P and lP^* in Λ_P^*, where P and lP^* are from the example above.



Now by Theorem 2.2.0.7, $\Lambda_{lP^*}/l = \Lambda_P^*$. We know that vertices of P^* are of the form u_F/l for the vertices u_F of lP^*, and that Λ_P^* is generated by $(1,0),(0,1/l)$. So these vertices live in Λ_P^* and they are primitive in Λ_P^* since there is a vertex v in $V(P)$ for any vertex u_F/l of P^* such that $\langle v, u_F/l\rangle = 1$. Thus, by Proposition 2.1.0.3, P^* is a 1-reflexive loop, say Q^*, in Λ_P^*. Moreover, multiplying by l, Q^* is isomorphice to lP^* with respect to lattice $\Lambda_{lP^*} = l\Lambda_P^*$. □

The associated 1-reflexive loop Q with respect to Λ_P has the same number of boundary points as l-reflexive loop $P \subseteq N$.

Now we can explain how there was a map that took a 1-reflexive centrally symmetric hexagon to a l-reflexive centrally symmetric hexagon.

Corollary 2.2.0.2. *Let R be a set of representatives of all isomorphism classes of 1-reflexive polygons(loops) such that for any $Q \in R$, $(0,1) \in N$ is a vertex of Q and $(0,1) \in M$ is a vertex of Q^*. If P is l-reflexive, then there exists $Q \in R$ such that P is isomorphic to the image of Q under right multiplication by* $\begin{pmatrix} l & i \\ 0 & 1 \end{pmatrix}$ *for $0 < i < l$ coprime to l.*

Proof. First, we need to prove that such R is possible. We know that we can guarantee $(0,1)$ to be a vertex of 1-reflexive Q by some unimodular map(or observe that all 16 1-reflexive polygons have $(0,1)$ as the vertex). If $(0,1) \in int(F)$ for some facet F of P, then $u_F = (0,1)$. If $(0,1) \in V(F_1 \cap F_2)$ for facets F_1, F_2 of P, then $(0,1) \in conv\{u_{F_1}, u_{F_2}\}$ since both u_{F_1} and u_{F_2} have y-coordinate 1.

By Theorem 2.2.0.7 there is an isomorphism $\mathbb{Z}^2 \to \Lambda_P \subseteq \mathbb{Z}^2$ by right multiplying an integer 2x2 matrix $H^{'}$ of determinant l. $H^{'}$ is a matrix formed by a basis of Λ_P. By Corollary 2.2.0.1, $P = Q^{'}H^{'}$ for some 1-reflexive polygon(loop) $Q^{'}$. Then, we choose some $Q \in R$ so that $Q^{'} = QU^{'}$ for some unimodular U'. By the Hermite normal form theorem[14], there exists a unimodular U such that $H := U^{'}H^{'}U$ is of the form $\begin{pmatrix} d & i \\ 0 & l/d \end{pmatrix}$ for d a divisor of l and $0 \leq i < d$. So we have $P^{'} = PU = QH$. The vertex $(0,1)$ of Q is sent to the vertex $(0, l/d)$ of $P^{'}$ by H. Since $P^{'}$ is l-reflexive $(0, l/d)$ must be primitive, hence $l/d = 1$. Thus, $H = \begin{pmatrix} l & i \\ 0 & 1 \end{pmatrix}$, and P is isomorphic to the image of Q under the map H.

For the dual, one checks that lP^* is equal to the image of Q^* under the map $M := l(H^T)^-2 = \begin{pmatrix} 1 & 0 \\ -i & l \end{pmatrix}$. Let $g = gcd(l,i)$. Then, there exists unique integers j,k with $0 \leq j < l/g$ such that $-ji + kl = g$. Therefore, we can obtain a unimodular matrix $J := \begin{pmatrix} l/g & j \\ i/g & k \end{pmatrix}$ so that $M \cdot J = \begin{pmatrix} l/g & j \\ 0 & g \end{pmatrix} =: K$ is a Hermite normal. The vertex $(0,1)$ of Q^* is sent to the vertex $(0,g)$ of lP^* by K, which implies that $g = 1$ since vertices of lP^* are primitive. Therefore, lP^* is ismorphic to the image of Q^* under the matrix K. □

Note that each *l-reflexive loop* has a well-defined **winding number** $w(P) \in \mathbb{Z}$. The best way to think of this winding number is to imagine a person standing on the origin facing towards some starting point on the boundary of P and and rotate the body as the eyes traverse the boundary counter-clockwise until they come back to the starting point. The winding number is the number of complete rotations the person has made. For the

[14]For every integer matrix A there is a unique integer matrix H such that $H = AU$ for some unimodular U. A Hermite normal form has the following properties: it is upper triangular, the entries are positive, the entries right to the diagonals are smaller than the diagonals

example we've been looking at, the winding number is 1 for both P and lP^*.

Corollary 2.2.0.3. *Let P be an l-reflexive loop. Then the sum of the length of P and the length of lP^* equals $12w(P)$.*

Proof. It was shown in [4] that the corollary holds for 1-reflexive loops. We use the above two corollaries to prove when P is l-reflexive. Let Q be the 1-reflexive loop respect to the lattice Λ_P. Let $b_N(P)$ denote the number of boundary points of P. We have that $b_{\Lambda_P}(Q) = b_N(P)$, and $b_{\Lambda_P^*}(Q^*) = b_{l\Lambda_P^*}(lQ^*) = b_{\Lambda_{lP^*}}(lP^*) = b_M(lP^*)$. Hence, the corollary follows. □

Now we show a different proof of no even index for *l-reflexive polygons* by applying Corollary 2.2.0.2. The proof of Theorem 2.1.0.6 is applicable to *l-reflexive loops* but the following one is not.

Proof. Assume that the index l is even for a *l-reflexive polygon* P. Suppose that $u_{F_1} = (0,1)$ so that (a,l) and (b,l) are vertices of F_1. They are primitive, so a and b are odd. This implies that $\frac{(a,l)+(b,l)}{2}$ is a lattice point of F. Suppose that (b,l) is a vertex of F_2 as well. By definition we must have that, given $u_{F_2} = (p,q)$, $bp + lq = l$ which implies that p is even and q is odd. This causes the other vertex of F_2 to be $(odd, even)$, which causes u_{F_3} to be $(even, odd)$, and so on. Hence, any facet of P and lP^* contains an interior lattice point. By 2.2.0.2 the corresponding *1-reflexive polygon* Q and its dual Q^* must contain an interior lattice point for their facets. However, none of the 16 *1-reflexive polygons* display this behavior. Thus, there's no *l-reflexive polygons* of even index. □

2.3. Roots

Recall that the definition of *Ehrhart polynomial* of a d-polytope P was given as $L_P(t) := \#(tP \cap \mathbb{Z}^d)$. It was briefly mentioned that if P is convex with integer vertices, then $L_P(t)$ is a polynomial in t of degree d. When we consider L_P as a polynomial over $\mathbb{C}$ and study its roots, a surprising result arises. It was proven in [15] that if all roots of $L_P(t)$ have real part $-1/2$, then P is a reflexive polytope up to an unimodular translation.[15] The

[15]Note that this interesting behavior displays an instance of the famous Riemann Hypothesis

converse is not true for $d \geq 4$. However, it was shown in [16] that the following Lemma holds for any dimension d.

Lemma 2.3.0.1. *The lattice polytope P is reflexive (up to lattice translation) if and only if*

$$\sum_{i=1}^{d} z_i = -\frac{d}{2}$$

where $z_1, \dots, z_d \in \mathbb{C}$ are the roots of L_P.

Proof. We can write L_P as $L_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0$. It is known that $c_d = vol(P)$, $c_{d-1} = 1/2 vol(\partial P)$[16], and $c_0 = 1$, credit to Ehrhart. Hence

$$-\sum_{i=1}^{d} z_i = \frac{c_{d-1}}{c_d} = \frac{1}{2} \cdot \frac{vol(\partial P)}{vol(P)}$$

Also P is reflexive (up to lattice translation) if and only if $d \cdot vol(P) = vol(\partial P)$[13]. Thus, the lemma follows. □

Now we extend the results of 1-reflexive to l-reflexive of dimension 2.

Proposition 2.3.0.4. *With the exception of l-reflexive polygon that is isomorphic to the convex hull of $\{(-1,-1), (-1,2), (2,-1)\}$, if $z \in \mathbb{C}$ is a root of $L_P(t)$, then $Re(z) = -\frac{1}{2l}$.*

Proof. Since P is a *l-reflexive polygon*, $L_P(t) = c_2 t^2 + c_1 t + c_0$. c_2 corresponds to the area of P and c_1 to one half times the number of the boundary points of P. Using the Pick's Theorem, $A = I + \frac{B}{2} - 1$, gives us $L_P(1) = I + B = A + \frac{1}{2}B + 1$. So $L_P(t) = At^2 + \frac{1}{2}Bt + 1$, and we need to show that tP has area At^2 and Bt many boundary points.

Suppose P has a set of $\{x_1, \dots, x_b\}$ boundary points. We have shown before that the area of P is $A(P) = \frac{1}{2}\sum_{i=1}^{b} A_i$ where A_i is the determinant of x_i and x_{i+1}. Hence, $A(tP)$ is precisely At^2. Also $b = \sum_{i=1}^{b} gcd(x_i, x_{i+1})$ so that bt is the number of the boundary points of tP.

Thus, for a *l-reflexive polygon* P we have that

$$L_P(t) = \frac{lb}{2}t^2 + \frac{b}{2}t + 1$$

[16] $vol(P) = \lim_{t\to\infty} \frac{1}{t^d} L_P(t) = c_d$ and $vol(\partial P) = \lim_{t\to\infty} \frac{1}{t_{d-1}}(L_P(t) - L_{P^\circ}(t))$

since the area of l-reflexive polygon is $\frac{lb}{2}$.

Let $z \in \mathbb{C}$ be a root of $L_P(t)$. By the quadratic formula

$$z = -\frac{1}{2l} \pm \frac{\sqrt{b^2 - 8lb}}{2lb}$$

$b^2 - 8lb \leq 0$ for all $3 \leq b \leq 9$ (b cannot be bigger than 9 since 1-reflexive can have at most 9 boundary points) and $l \geq 1$, except when $b = 9, l = 1$, and the convex hull of $\{(-1,-1),(-1,2),(2,-1)\}$ is the only *1-reflexive polygon* with 9 boundary points, up to isomorphism. □

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