UNIVERSITY OF OREGON

Higher Order Beliefs and Sequential Reciprocity

by

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A thesis submitted in partial fulfillment for the undergraduate honor

in the
Jiabin Wu
Mathematics

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Declaration of Authorship

I, Lianjie Jiang, declare that this thesis titled, 'Higher Order Beliefs and Sequential Reciprocity' and the work presented in it are my own. I confirm that:

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- Where any part of this thesis has previously been submitted for the honor or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
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Signed: ______________________________

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"Like all of mathematics, game theory is a tautology whose conclusions are true because they are contained in the premises."

Thomas Flanagan (1998)

UNIVERSITY OF OREGON

Abstract

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This paper discusses the mathematics behind psychological game theory. We studies the modeling of reciprocity in sequential games and we argue that whether higher order beliefs should be updated in a game depends on the structure of the game. To incorporate this novel feature, we propose a refinement of the solution concept, *sequential reciprocity equilibrium*, developed by Dufwenberg and Kirchsteiger (2004) and apply it to an example to demonstrate its applicability.

Acknowledgements

I really thank my research supervisor, Jiabin Wu. Without his assistance and dedicated involvement in every step throughout the process, this paper would have never been accomplished.

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Abbreviations

NE	**N**ash **E**quilibrium
SPE	**S**ubgame **P**erfect **E**quilibrium
SRE	**S**equential **R**eciprocity **E**quilibrium

To my family

Chapter 1

Introduction

Game theory is "*the study of mathematical models of conflict and cooperation between intelligent rational decision-makers.*"(Myerson, 1991). Game theory is widely applied in economics, political science, and psychology, as well as logic, computer science and biology. The main reason we are interested in game theory is that it can be applied in many ways in our daily lifes. Why do people always leave work to do until the last minute? This could be analyzed by game theory. "The present you" is playing games with "the future you". "The present you" evaluates that during the weekend, it is more valuable to spend time on playing video games or going to a party, "the future you" prefers to start doing homework at the weekend. But it is "the present you"'s turn, you went to party and get drunk, and you in the future will just regret the decisions you have made at past and work until 3am.

1.1 History and Impacts of Game Theory

Back into history, the earliest example of formal game-theoretic analysis is presented by Antoine Cournot to study duopoly in 1838. In 1921, French mathematician Emile Borel suggested a formal theory of games. Based on this, Hungarian-American mathematician John von Neumann improved his work with "theory of parlor games" in 1928. After 1944, game theory was established as a field in its own right from von Neumann and the economist Oskar Morgenstern's book "Theory of Games and Economic Behavior".

In 1950, John Nash proved the existence of "Nash Equilibrium" in his PhD. thesis "non-cooperative games", which is, a finite game always has an equilibrium point when all players choose actions which are best for them given their opponents' choices. Many mathematicians and economists have started to conduct research on noncooperative games since then. In the 1950s and 1960s, game theory was broadened theoretically and applied to problems of war and politics. Since the 1970s, it has driven a revolution in economic theory. Moreover, it has also been applied widely in sociology and psychology, and established links with evolution and biology. In 1994, the Nobel prize in economics awarded to Nash, John Harsanyi, and Reinhard Selten for their contributes on game theory. (Theodore L. Turocy, and Bernhard von Stengel, 2001)

1.2 Psychological Game Theory

Psychological game theory is relatively a new area in game theory initated by Geanakoplos et al. (1989) and further developed by Battigalli and Dufwenberg (2009). Rabin (1993) proposes an influential model based on psychological game theory which models peoples' reciprocal behavior (act kindly (unkindly) in response to kind (unkind) actions) in strategic interactions. Rabin proposes a solution concept called "fairness equilibrium" that can be applied to all normal form games. Dufwenberg and Kirchsteiger (2004) apply the same techniques to extensive form games and proposes a new solution concept called "sequential reciprocity equilibrium". In this paper, we will reinvestigate Dufwenberg and Kirchsteiger's model and make a novel improvement on the sequential reciprocity equilibrium to make it better capture the results of some psychological game experiments.

1.3 Mathematical Tools Being used in Psychological Game Theory

To prove the existence of a sequential reciprocity equilibrium, two key theorems are needed. Here we provide an introduction to the theorems.

Definition 1.1. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two topological spaces. A ***correspondence*** Γ is a subset of $X \times Y$ with the property that for every $x \in X$ there exists $y \in Y$ such that $(x, y) \in \Gamma$. Formally, it is written as

$$\Gamma : X \rightrightarrows Y$$

which assigns to each $x \in X$ a non-empty subset $\Gamma(x) = \{y \in Y \mid (x, y) \in \Gamma\}$ of Y,

Definition 1.2. Given a correspondence $\Gamma : X \rightrightarrows Y$ and a subset E of Y,

1. the ***upper-inverse image*** of E is

$$\Gamma_+^{-1}[E] = \{x \in X \mid \Gamma(x) \subseteq E\}$$

.

2. the ***lower-inverse image*** of E is

$$\Gamma_{-}^{-1}[E] = \{x \in X \mid \Gamma(x) \cap E \neq \emptyset\}$$

.

Definition 1.3. A correspondence $\Gamma : X \rightrightarrows Y$ is said to be

1. ***upper hemicontinuous*** at $x \in X$ provided for all open sets O, such that $O \supseteq \Gamma(x)$, the set $\Gamma_{+}^{-1}[O]$ is open in X.
2. ***lower hemicontinuous*** at $x \in X$ provided for all open sets O, such that $O \cap \Gamma(x) \neq \emptyset$, the set $\Gamma_{-}^{-1}[O]$ is open in X.
3. ***continuous*** at $x \in X$ provided it is both upper hemicontinuous and lower hemicontinuous.

If Γ is continuous/upper hemicontinuous/lower hemicontinuous at every $x \in X$ then Γ is said to be continuous/upper hemicontinuous/lower hemicontinuous.

From here on it will be assumed that topologies X and Y are metrizable.

Definition 1.4. A correspondence $\Gamma : X \rightrightarrows Y$ is ***compact-valued*** if for all $x \in X$, $\Gamma(x)$ is compact in the topology on Y.

1.3.1 Berge's maximum theorem

Berge's maximum theorem dictate properties for correspondence *argmax*. (maximization being the common link amongst all of economics)

Theorem 1.5. *Suppose that $f : X \times A \to \mathbb{R}$ is a continuous function where $X \subseteq \mathbb{R}^l$ and $A \subseteq \mathbb{R}^m$. Suppose in addition that that any correspondence $X^f : A \rightrightarrows X$ is compact-valued and continuous. If the correspondence $X^O : A \rightrightarrows X$ is defined as*

$$X^O(\alpha) = \underset{x \in X^f(\alpha)}{\operatorname{argmax}} f(x, \alpha),$$

then X^O is upper hemicontinuous.

(Claude Berge, 1963)

1.3.2 Katutani's fixed point theorem

Theorem 1.6. *Let $X \subset \mathbb{R}^n$ be a non-empty, compact and convex set. If $\Gamma : X \rightrightarrows X$ is a well defined ($\Gamma(x) \neq \emptyset$), convex-valued, and upper hemicontinuous then Γ has a fixed point. i.e. there exists $x^* \in X$ such that $x^* \in \Gamma(x^*)$.*

(For proof, see [1, p. 6-9] by Anton Badev and Matthew Hoelle)

Chapter 2

Sequential Games

In game theory, there are two basic types of games we are interested in. Normal form games model situations in which all players make decisions simultaneously. In sequential games, players take turns and make decisions sequentially. In this paper, we mainly focus on sequential games.

2.1 The structure of a sequential game

To define a sequential game, let $N = \{1, \cdots, n\}$ be the set of players where $n \geq 2$. Let H be the set of choice profiles, or histories. Let A_i be the set of behavior strategies of $i \in N$; each strategy assigns for each history $h \in H$ a probability distribution on the set of possible choices of i at h. Note that A_i includes mixed strategies. A mixed strategy is a randomization among a player's choices. Define $A = \prod_{i \in N} A_i$. Using the assignment of payoffs to the end nodes we can derive a payoff function for each player which depends on what strategy profile in A is played. Let $\pi_i : A \to R$ denote the payoff function. We shall refer to π_i as player $i's$ material payoff function. The material payoffs represent money, or some other objectively measurable quantities. A player's utility function may not be identical to the material payoff he or she receives. A player may have concerns beyond his/her own material payoff, be affected by emotions, psychological biases and etc. In this paper, we assume that players are reciprocal, meaning that they gain additional utilities (reciprocal payoff) by reciprocating others in addition to their own material payoffs. In Section 2.2 we provide several examples for illustrations.

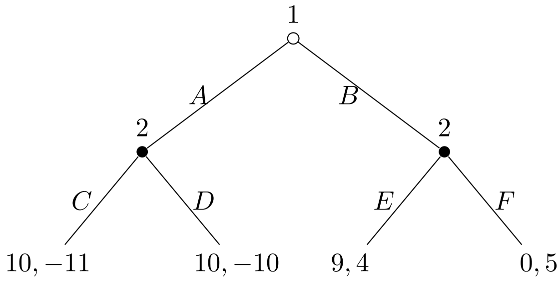

FIGURE 2.1: Game 1

2.2 Examples

In game 1 displayed above, assume that players are self-interested and we apply backward induction to solve the game. Backward induction is the process of reasoning backwards in time, from the end of the game, to determine a sequence of optimal actions. Starting from player 2, he chooses an action at each decision node that grants the highest material payoff for himself, which is D, F, respectively. Given this, player 1 will choose A to get the highest material payoff for herself. Hence, there is a unique subgame perfect equilibrium $(A, (D, F))$ which leads to the outcome of $(10, -10)$. This subgame perfect equilibrium may not capture what people would actually behave in the real life for the following reasons: Suppose player 1 cares about player 2's well-being and she does not want player 2 to get -10. Then she would kindly choose B to avoid reaching the outcomes of $(10, -11)$ and $(10, -10)$. Now it is player 2's turn. He looks at the history and observes that player 1 has already chosen

B. He knows that the best choice for player 1 is to choose A, which will gives him either -10 or -11. However player 1 did not choose A. Instead, player 1 chose B to sacrifice her own payoff. Since player 2 now knows that player 1 is kind to him by choosing B, as a return, player 2 may choose E to give player 1 a higher payoff. Hence, $(B,(D,E))$ is a potential "psychological equilibrium". There are more psychological equilibria in this game. If player 1 does not care about player 2's payoff and player 2 doesn't care about player 1 as well, the psychological equilibrium would agree with the subgame perfect equilibrium $(A,(D,F))$. If player 1 does not care about player 2's payoff and player 2 is reciprocal, the psychological equilibrium may be $(A,(D,E))$. If instead player 1 cares about player 2's payoff and player 2 does not want to pay back, the psychological equilibrium would be $(B,(D,F))$.

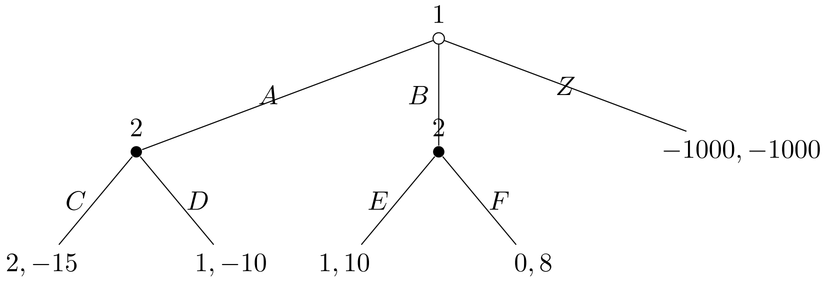

FIGURE 2.2: Game 2

In game 2, one can observe that player 2 really wants to avoid a payoff of -1000. If player 1 chooses A, should player 2 be thankful? The answer is probably no, Choosing A from $\{A, B\}$ does not imply that player 1 cares about player 2's well-being, it just tells us player 1 want to avoid -1000 for herself as well. This example tells us one needs to toss out extreme strategies (Dufwenberg and Kirchsteiger, 2004).

2.3 The Kindness Functions

In the games we discussed above, to obtain i's utility, which is the function that i wishes to maximize, we shall add a reciprocity payoff to i's utility function besides the material payoff. The reciprocity payoff depends on i's beliefs about other players' strategies and beliefs. We represent beliefs as behavior strategies. However, in order to avoid confusion, we introduce separate notation for beliefs. Let $B_{ij} = A_j$ be the set of possible beliefs of player i about the strategies of player j. Let $C_{ijk} = B_{jk} = A_k$ be the set of possible beliefs of player i about the beliefs of player j about the strategies of player k.

From game 1, player's kindness and perception of another player's kindness may differ after different histories. To capture this, it is important to track how each players's behavior, beliefs, kindness, and perception of others' kindness differ across histories. What Dufwenberg and Kirchsteiger(2004) did is as follows: Let Γ be a finite-stage games with observed actions and without nature. With $a_i \in A_i$, $h \in H$, let $a_i(h)$ be the (updated) strategy that prescribes the same choices as a_i, except for the choices that define history h which are made with probability 1. Note $a_i(h)$ is uniquely defined for any history. For beliefs $b_{ij} \in B_{ij}$ or $c_{ijk} \in C_{ijk}$, define updated beliefs $b_{ij}(h)$ and $c_{ijk}(h)$ in a fashion completely analogous to $a_i(h)$. We shall later show that Dufwenberg and Kirchsteiger(2004)'s definition on higher order beliefs may be problematic in certain applications.

Also from game 2, we know that "extreme" strategies need to be tossed out. We apply the following definition:

Definition 2.1. Player i's efficient strategies are

$$\begin{aligned} E_i = \{ & a_i \in A_i \mid \text{there exists no } a_i' \in A_i \text{ such that for all } h \in H, \\ & (a_j)_{j \neq i} \in \prod_{j \neq i} A_j, \text{ and } k \in N \text{ it holds that} \\ & \pi_k\big(a_i'(h), (a_j(h))_{j \neq i}\big) \geq \pi_k\big(a_i(h), (a_j(h))_{j \neq i}\big), \\ & \text{with strict inequality for some } \big(h, (a_j)_{j \neq i}, k\big)\}. \end{aligned}$$

Then we can define what players are expected to get if they make certain decisions.

Definition 2.2. Player j's belief about player i's equitable payoff is

$$\begin{aligned} \pi_j^{e_i}\big((b_{ij})_{j \neq i}\big) = \frac{1}{2} \cdot [& \max\{\pi_j\big(a_i, (b_{ij})_{j \neq i}\big) \mid a_i \in A_i\} \\ & + \min\{\pi_j\big(a_i, (b_{ij})_{j \neq i}\big) \mid a_i \in E_i\}], \end{aligned}$$

This function provides a norm for player j to evaluate i's kindness towards to him. If player i chooses a strategy a_i such that $\pi_j(a_i, (b_{ij})_{j \neq i}) = \pi_j^{e_i}((b_{ij})_{j \neq i})$, then his kindness to j is zero. Otherwise i's kindness to j is proportional to how much more or less material payoff than $\pi_j^{e_i}((b_{ij})_{j \neq i})$ that i thinks will be the consequence for j.

Definition 2.3. The kindness of player i to another player j at history $h \in H$ is given by the function $\kappa_{ij} : A_i \times \prod_{j \neq i} B_{ij} \to \mathbb{R}$

$$\kappa_{ij}\Big(a_i(h), \big(b_{ij}(h)\big)_{j \neq i}\Big) = \pi_j\Big(a_i(h), \big(b_{ij}(h)\big)_{j \neq i}\Big) - \pi_j^{e_i}\Big(\Big(b_{ij}(h)\Big)_{j \neq i}\Big).$$

Intuitively, this definition means that the i's kindness to j's is proportional to "the size of gift".

Definition 2.4. Player i's beliefs about how kind player $j \neq i$ is to i at history $h \in H$ is given by the function $\lambda_{iji} : B_{ij} \times \prod_{k \neq j} C_{ijk} \to \mathbb{R}$ defined by

$$\lambda_{iji}\Big(b_{ij}(h), \big(c_{ijk}(h)\big)_{k \neq j}\Big) = \pi_i\Big(b_{ij}(h), \big(c_{ijk}(h)\big)_{k \neq j}\Big) - \pi_i^{e_j}\big(\big(c_{ijk}(h)\big)_{k \neq j}\big).$$

In most cases, $B_{ij} = A_j$ and $C_{ijk} = B_{jk}$, the function λ_{iji} is mathematically equivalent to κ_{ji}.

The utility function is the objective function that a player wishes to maximize:

Definition 2.5. Player i's utility at history $h \in H$ is the function

$$U_i : A_i \times \prod_{j \neq i} \left(B_{ij} \times \prod_{k \neq j} C_{ijk} \right) \to \mathbb{R}$$

, defined by

$$U_i\big(a_i(h), \big(b_{ij}(h), \big(c_{ijk}(h)\big)_{k\neq j}\big)_{j\neq i}\big)$$

$$= \pi_i\big(a_i(h), \big(b_{ij}(h)\big)_{j\neq i}\big)$$

$$+ \sum_{j\in\mathbb{N}\setminus\{i\}} \Big(Y_{ij} \cdot \kappa_{ij}\big(a_i(h), \big(b_{ij}(h)\big)_{j\neq i}\big) \cdot \lambda_{iji}\big(b_{ij}(h), \big(c_{ijk}(h)\big)_{k\neq j}\big)\Big),$$

where Y_{ij} is an exogenously given non-negative number for each $j \neq i$. The constant Y_{ij} measures how sensitive i is to reciprocity concerns regarding player j.

Now we can solve the maximization problem of a player's utility. Each players utility is the sum of his material payoff and his reciprocity payoff respect to all other players.

Dufwenberg and Kirchsteiger(2004) proposes the following solution concept for solving sequential game with reciprocal players:

Definition 2.6. The profile $a^* = (a_i^*)_{i\in N}$ is a sequential reciprocity equilibrium (SRE) if for all $i \in N$ and for each history $h \in H$ it holds that

1. $a_i^*(h) \in \text{argmax}_{a_i\in A_i(h,a^*)} U_i(a_i, (b_{ij}(h), (c_{ijk}(h))_{k\neq j})_{j\neq i})$,
2. $b_{ij} = a_j^*$ for all $j \neq i$,
3. $c_{ijk} = a_k^*$ for all $j \neq i, k \neq j$

In next chapter, we will argue that not every beliefs should be updated, and some updated beliefs should not be used for solving the sequential reciprocity equilibrium.

Chapter 3

Analysis

The main difference between this paper and Dufwenberg and Kirchsteiger(2004) is that we argue that some higher-order beliefs should not be updated. To see this, we start by introducing an interesting sequential game.

3.1 A three player sequential game

This game is developed by my advisor, Jiabin Wu, in his experimental study. First, let us look at a simple version of the game. Player 1 moves first to make a decision between left and right. If he chooses Left, each player gets a payoff of 5. If he chooses right, player 2 will make a move to choose between left and right. If she chooses left, the

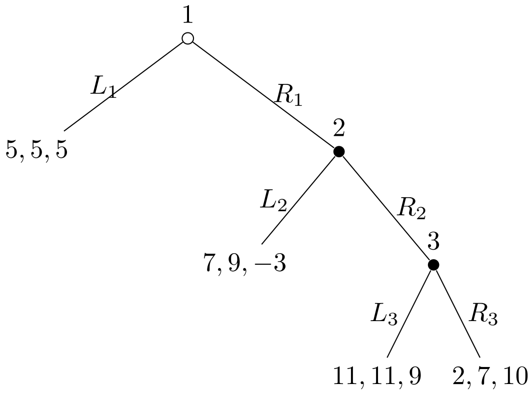

FIGURE 3.1: Game 3

outcome for the three players are $7, 9, -3$ respectively. If she chooses right, player 3 could choose between left resulting in an outcome of $11, 11, 9$ or right resulting in a outcome of $2, 7, 10$. Let me explain briefly what happened during the experiment.

In this experiment, whenever player 3 is making a choice, it means that player 1 and 2 have already chosen right. Player 3 knows that player 2 is kind to him, since if player 2 chooses left player 3 will get a payoff of -3. Yet, player 2 ends up choosing right to give player 3 a payoff of either 9 or 10. Player 3 thus has an incentive to reciprocate player 2. On the other hand, although player 1 did choose right, player 3 does not know if player 1 is kind to him or not. If he believes that players 1's belief about player 2 (c_{312}) is left, then he believes that player 1 is unkind to him because player 1 expected player 3 to get playoff -3. If he instead believes players 1's belief about player 2 (c_{312}) is right, then he believes that player

1 is kind to him since he will get 9 and 10. If player 3 wants to take revenge to player 1, he may choose right. Otherwise, he may choose left. Suppose we use Dufwenberg and Kirchsteiger's definition of sequential reciprocity equilibrium. From definition 2.6, we know that all the beliefs should be updated. In this case, player 3 should always choose left to repay player 1 and 2. But the point is, when player 1 making decision, he does not know player 2's action, so it makes no sense for player 3 to update player 1's belief about player 2 on player 2's future move. In the experiment, most participants in the role of player 3 actually chose Right to take revenge on player 1. We give a brief experiment result at the end of this chapter, for detials, see [6, Jiabin Wu 2017].

3.2 Solve for sequential reciprocity equilibrium (SRE)

Let $N = \{1, 2, 3\}$. Since only player 3's behavior matters, for simplicity, let $Y_{31} = 1$, and all other $Y_{ij} = 0$ for $i \neq j$. Assume player 3 gets to make a choice. In this case, $a_1(h) = R, a_2(h) = R$. Suppose all beliefs are updated, as in Dufwenberg and Kirchsteiger(2004) $b_{12}(h) = R$, $c_{312}(h) = R$. We first compute player 1's belief about player 3's "equitable payoff":

$$\pi_3^{e_1}(b_{12}, b_{13}) = \frac{1}{2} \cdot [\max\{\pi_3(a_1, (b_{12}, b_{13})) \mid a_1 \in A_1\} + \min\{\pi_3\big(a_1, (b_{12}, b_{13})\big) \mid a_1 \in E_1\}].$$

Since we don't know what player 3 will choose, let's assume player 3 choose mix strategy $b_{13} = a_3 = \mu_3 \cdot L + (1 - \mu_3) \cdot R$ for $\mu_3 \in [0, 1]$ (play L with probability μ_3). Then we have,

$$\begin{aligned}\pi_3^{e_1}(b_{12}, b_{13}) &= \pi_3^{e_1}(R, (\mu_3 \cdot L + (1 - \mu_3) \cdot R)) \\ &= \frac{1}{2} \cdot (\mu_3 \cdot 9 + (1 - \mu_3)10 + 5) \\ &= 7.5 - \frac{\mu_3}{2}.\end{aligned}$$

Next we compute what's player 3's belief about player 1's kindness to him.

$$\begin{aligned}\lambda_{313}\Big(b_{31}(h), c_{312}(h), c_{313}(h)\Big) &= \pi_3\Big(b_{31}(h), c_{312}(h), c_{313}(h)\Big) \\ &\quad - \pi_3^{e_1}\Big((c_{312}(h), c_{313}(h))\Big) \\ &= \pi_3(R, R, \mu_3 \cdot L + (1 - \mu_3) \cdot R) \\ &\quad - \pi_3^{e_1}\Big(R, \mu_3 \cdot L + (1 - \mu_3) \cdot R\Big) \\ &= \mu_3 \cdot 9 + (1 - \mu_3)10 - (7.5 - \frac{\mu_3}{2}) \\ &= 2.5 - \frac{\mu_3}{2}.\end{aligned}$$

Since $\mu_3 \in [0,1]$, $\kappa_{13} > 0$, one can tell that player 1 is kind to player 3.

Next we want to figure out the kindness of player 3 to player 1.

$$\begin{aligned}
\kappa_{31}\Big(a_3(h),(b_{31}(h)),b_{32}(h)\Big) &= \pi_1\Big(a_3(h),b_{31}(h),b_{32}(h)\Big) - \pi_1^{e_3}\Big((b_{31}(h),b_{32}(h))\Big) \\
&= \pi_1(\mu_3 \cdot L + (1-\mu_3)\cdot R, R, R) - \pi_1^{e_3}\Big(R,R\Big) \\
&= \mu_3 \cdot 11 + (1-\mu_3)2 - (\frac{11+2}{2}) \\
&= 9\mu_3 - 4.5.
\end{aligned}$$

Finally, player 3's utility by choosing $\mu_3 \cdot L + (1-\mu_3)\cdot R$ is

$$\begin{aligned}
&U_3\big(\mu_3 \cdot L + (1-\mu_3)\cdot R, \big(b_{ij}(h), \big(c_{ijk}(h)\big)_{k\neq j}\big)_{j\neq i}\big) \\
&= U_3(\mu_3 \cdot L + (1-\mu_3)\cdot R, R, R) \\
&= \pi_i\big(a_i(h), \big(b_{ij}(h)\big)_{j\neq i}\big) \\
&+ \sum_{j\in\mathbb{N}\backslash\{i\}} \Big(Y_{ij}\cdot \kappa_{ij}\big(a_i(h),\big(b_{ij}(h)\big)_{j\neq i}\big)\cdot \lambda_{iji}\big(b_{ij}(h),\big(c_{ijk}(h)\big)_{k\neq j}\big)\Big) \\
&= (\mu_3 \cdot 9 + (1-\mu_3)\cdot 10) + (1\cdot(9\mu_3 - 4.5)\cdot(2.5 - 0.5\mu_3)) \\
&= -4.5\mu_3^2 + 23.75\mu_3 - 1.25.
\end{aligned}$$

Take derivative with respect to μ_3 we get $U_3' = -9\mu_3 + 23.75$. Set $U_3' = 0$, we find $\mu_3 = 95/36$ which is not in the domain, so the boundary points($\mu = 0,1$) are the maximum and the minimum. When player 3 chooses L, $\mu_3 = 1, U_3 = 18$.

When player 3 chooses R, $\mu_3 = 0, U_3 = -1.25$.

So the best response for player 3 is L.

Since we assume $Y_{ij} = 0$ for $i \neq j$ except $Y_{31} = 1$. Player 1 and 2 should maximize their material payoffs. By backward induction, the SRE according to Dufwenberg and Kirchsteiger(2004) is (R, R, L).

3.3 Discussion

From the calculation we did, we can see the player 3 believes that player 1 is kind to him by choosing right. But in the experiment most games end with (R, L) or (R, R, R). Then we back to the discussion at the beginning of this chapter, we believe player 3 is not certain if player 1 is kind or unkind to him by choosing R. The reason we solve SRE to be (R, R, L) is by the definition of SRE in Dufwenberg and Kirchsteiger(2004), we update player 3's belief about player 1 about player 2(c_{312}). But it may not be the case in this game.

3.4 Updating Rules for Higher-order Beliefs

To determine which higher-order beliefs should be updated, one should think why we need to update the beliefs. In Game 3, the reason we should not use updated belief c_{312} is that when player 1 is making choice, his kindness to player 3 depends on player 2's future move. This implies that beliefs about future move should not be updated.

In a three player game, let us assume $N = \{1, 2, 3\}$. Consider a game with the same structure as game 3: player 1 moves first, player 2 moves second, then player 3 moves at the end. Each player only has 1 choice with 2 options. Then all the first-order beliefs are listed in the following,

$$B_{ij} = \{b_{12}, b_{13}, b_{21}, b_{23}, b_{31}, b_{32}\}.$$

In the definition of updating beliefs, we know that b_{12}, b_{13}, b_{23} should not be updated. And b_{21}, b_{31}, b_{32} should be updated, and this is reasonable, since when player 2 and 3 is making choices, they already see what happened in the past. All the second-order beliefs are listed in the following,

$$C_{ijk} = \{c_{123}, c_{132}, c_{213}, c_{231}, c_{312}, c_{321}\}.$$

By the definition of updating beliefs, $c_{123}, c_{132}, c_{213}$ should not be updated. All these three beliefs are predictions of future decisions, Hence, we should not update them. On the other hand, c_{231}, c_{321}

should be updated since both player 2 and player 3 can observe player 1's decision when they are making decisions. The only left is c_{312}, and we have already showed that this belief should not be updated.

we give a formal definition on belief-updating rules as follows:

Definition 3.1. In a finite player multi-stage games with observes actions and without nature. Define $N = 1, 2, 3 \cdots$, for all $i < j$, player i moves before player j, For C_{ijk}, $c_{kij}(h) = c_{kij} \neq a_j$ for all $i < j < k$.

3.5 Solve for SRE using the new definition on belief-updating rules

In Game 3, let $N = \{1, 2, 3\}$. Assume $Y_{31} = 1$, and all other $Y_{ij} = 0$ for $i \neq j$. Assume player 3 gets to make choices, i.e. $a_1(h) = R, a_2(h) = R$. All the beliefs expect c_{312} should be updated, i.e. $b_{12}(h) = R$, $c_{312}(h) = c_{312}$. Assume that player 3 chooses a mixed strategy $b_{13} = a_3 = \mu_3 \cdot L + (1 - \mu_3) \cdot R$ for $\mu_3 \in [0, 1]$ (play L with probability μ_3). Let $c_{312} = \mu_2 \cdot L + (1 - \mu_2) \cdot R$. Let us compute player 1's belief about player 3's "equitable payoff":

$$\pi_3^{e_1}(b_{12}, b_{13}) = \frac{1}{2} \cdot [\max\{\pi_3(a_1, (b_{12}, b_{13})\big) \mid a_1 \in A_1\} \\ + \min\{\pi_3\big(a_1, (b_{12}, b_{13})\big) \mid a_1 \in E_1\}].$$

More specifically,

$$\begin{aligned} \pi_3^{e_1}(b_{12}, b_{13}) &= \pi_3^{e_1}(\mu_2 \cdot L + (1-\mu_2) \cdot R, \mu_3 \cdot L + (1-\mu_3) \cdot R) \\ &= \frac{1}{2} \cdot (\max\{5, \mu_2 \cdot (-3) + (1-\mu_2)\big(\mu_3 \cdot 9 + (1-\mu_3)10\big)\} \\ &+ \min\{5, \mu_2 \cdot (-3) + (1-\mu_2)\big(\mu_3 \cdot 9 + (1-\mu_3)10\big)\}) \\ &= \frac{1}{2} \cdot (5 + \mu_2 \cdot (-3) + (1-\mu_2)\big(\mu_3 \cdot 9 + (1-\mu_3)10)). \end{aligned}$$

In an SRE, beliefs should be consistent with choices: $c_{312}(h) = c_{312} = b_{12} = a_2$. We have

$$\begin{aligned} \lambda_{313}\Big(b_{31}(h), c_{312}(h), c_{313}(h)\Big) &= \pi_3\Big(b_{31}(h), c_{312}(h), c_{313}(h)\Big) \\ &- \pi_3^{e_1}\Big((c_{312}(h), c_{313}(h))\Big) \\ &= \pi_3(R, \mu_2 \cdot L + (1-\mu_2) \cdot R, \mu_3 \cdot L + (1-\mu_3) \cdot R) \\ &- \pi_3^{e_1}\Big(\mu_2 \cdot L + (1-\mu_2) \cdot R, \mu_3 \cdot L + (1-\mu_3) \cdot R\Big) \\ &= \mu_2 \cdot (-3) + (1-\mu_2)\big(\mu_3 \cdot 9 + (1-\mu_3)10\big) \\ &- (\frac{1}{2} \cdot (5 + \mu_2 \cdot (-3) + (1-\mu_2)\big(\mu_3 \cdot 9 + (1-\mu_3)10\big) \\ &= -2.5 + \frac{1}{2}(\mu_2 \cdot (-3) + (1-\mu_2)(\mu_3 \cdot 9 + (1-\mu_3)10 \end{aligned}$$

Next we want to figure out the kindness function of player 3 to player 1.

$$\begin{aligned}\kappa_{31}\Big(a_3(h),(b_{31}(h)),b_{32}(h)\Big) &= \pi_1\Big(a_3(h),b_{31}(h),b_{32}(h)\Big)-\pi_1^{e_3}\Big((b_{31}(h),b_{32}(h))\Big)\\ &= \pi_1(\mu_3\cdot L+(1-\mu_3)\cdot R,R,R)-\pi_1^{e_3}\Big(R,R\Big)\\ &= \mu_3\cdot 11+(1-\mu_3)2-(\frac{11+2}{2})\\ &= 9\mu_3-4.5.\end{aligned}$$

Finally, player 3's utility by choosing $\mu_3 \cdot L + (1-\mu_3)\cdot R$ is

$$\begin{aligned}&U_3\big(\mu_3\cdot L+(1-\mu_3)\cdot R,\big(b_{ij}(h),\big(c_{ijk}(h)\big)_{k\neq j}\big)_{j\neq i}\big)\\ &= U_3(\mu_3\cdot L+(1-\mu_3)\cdot R,R,R)\\ &= \pi_i\big(a_i(h),\big(b_{ij}(h)\big)_{j\neq i}\big)\\ &+\sum_{j\in\mathbb{N}\setminus\{i\}}\Big(Y_{ij}\cdot\kappa_{ij}\big(a_i(h),\big(b_{ij}(h)\big)_{j\neq i}\big)\cdot\lambda_{iji}\big(b_{ij}(h),\big(c_{ijk}(h)\big)_{k\neq j}\big)\Big)\\ &= (\mu_3\cdot 9+(1-\mu_3)\cdot 10)\\ &+(1\cdot(9\mu_3-4.5)\cdot\big(-2.5+\frac{1}{2}(\mu_2\cdot(-3)+(1-\mu_2)(\mu_3\cdot 9+(1-\mu_3)10))\big)\\ &= -1.25+23.75\mu_3-4.5\mu_3^2+29.25\mu_2-60.75\mu_3\mu_2+4.5\mu_3^2\mu_2.\end{aligned}$$

Take derivative with respect to μ_3, we have $\frac{\partial U_3}{\partial \mu_3} = -9\mu_3 + 23.75 - 60.75\mu_2 + 9\mu_3\mu_2$. Set $\frac{\partial U_3}{\partial \mu_3} = 0$, we obtain $\mu_3 = \frac{23.75-60.75\mu_2}{9-9\mu_2} = 6.75 + \frac{37}{9\mu_2-9}$ as long as $\mu_2 \neq 1$. Take the second derivative with respect to μ_3, we get $\frac{\partial^2 U_3}{\partial \mu_3^2} = -9 + 9\mu_2 < 0$, this demonstrates that the interior solution is a maximum point. The best response depends on μ_2. We have three cases here, first case is $\mu_2 \in (0, 1)$, which means player 2 is taking a mixed strategy, implying that $U_2(L) = U_2(R)$, this leads to $\mu_3 = 0.5$. And since $\mu_3 = 0.5$ is the best response for player 3, we could set $0.5 = 6.75 + \frac{37}{9\mu_2 - 9}$, solve for μ_2, we get $\mu_2 = 0.342$. By backward induction, $U_1(L) = 5 < 6.671 = U_1(R)$, so player 1 will choose R. Our mixed SRE here is $(R, (0.342L + 0.658R), (0.5L + 0.5R))$.

The second case is $\mu_2 = 0$, i.e. player 2 chooses R. Then $\frac{\partial U_3}{\partial \mu_3} = -9\mu_3 + 23.75 > 0$, hence the best response for player 3 is $\mu_3 = 1$, i.e. player 3 chooses L. And when player 3 chooses L, player 2's best response is R, so all the players are playing their best responses, we have a pure SRE (R, R, L).

The third case is $\mu_2 = 1$, i,e, player 2 chooses L. Then $\frac{\partial U_3}{\partial \mu_3} = -37$, hence the best response for player 3 is $\mu_3 = 0$, i.e. player 3 chooses R. And when player 3 chooses R, player 2's best response is L, so all the players are playing their best responses, we have a pure SRE (R, L, R).

This solution makes more sense, the more player 3 believes player 1 believes player 2 chooses L (the larger the μ_2), the more likely he will choose R, because player 1 is very unkind to player 3 by believing that player 2 would choose L. In sum, there are three SREs in this game. A mixed one $(R, (0.342L + 0.658R), (0.5L + 0.5R))$ and two pure ones $(R, L, R), (R, R, L)$.

In Prof. Wu's experiment, 131 of 144 (91%) Player 1 chose Right. Given this, 55 of 131 (42%) Player 2 chose Right, and this was followed by 24 of 55 (44%) Player 3 choosing Left, which is pretty consistent with the mixed SRE we find.

Appendix A

Proof of Existence of the Sequential Reciprocity Equilibrium

Dufwenberg and Kirchsteiger(2004) developed the solution concept of the Sequential Reciprocity Equilibrium and as well gave a elegant proof of the existence. We only did a small improvment of the theorem but the existence still holds. So let me quote their proof here. To prove the existence, the key theorems are Berge's Maximum Theorem and Katutani's fixed point theorem, which we introduced before.

Proof. Let $X_i(h)$ be i's set of (possibly randomized) choices at history

$h \in H$. If $x \in X_i(h)$, let $a_i(h) \backslash x$ be the strategy of i's which specifies the choice x at h, but which is otherwise just like $a_i(h)$. Define correspondences $\beta_{i,h} : A \to X_i(h)$ and $\beta : A \to \prod_{(i,h) \in N \times H} X_i(h)$ by

$$\beta_{i,h}(a) = \underset{x \in X_i(h)}{\operatorname{argmax}} \, U_i\big(a_i(h) \setminus x, \big(a_j(h), \big(a_k(h)\big)_{k \neq j}\big)_{j \neq i}\big),$$

$$\beta(a) = \prod_{(i,h) \in N \times H} \beta_{i,h}(a).$$

The sets $\prod_{(i,h) \in N \times H} \beta_{i,h}(a)$ and A are topologically equivalent, so β is equivalent to a correspondence $\gamma : A \to A$ which is defined in the obvious way. Fixed points under γ are SREs. To see this, note that $\beta_{i,h}$ caters to condition (1) of Definition 2.6 plugging in the correct beliefs as mandated by conditions (2) and (3). Thus $\beta_{i,h}$ effectively finds the optimal strategies in $A_i(h, a)$, in conformance with (1), although this is here done using the optimal choices in $X_i(h)$. β and γ are combined best-response correspondences, and since γ is a correspondence from A to A it is amenable to fixed point analysis.

It remains to show that γ possesses a fixed point. Berge's maximum principle guarantees that $\beta_{i,h}$ is non-empty, closed-valued, and upper hemi-continuous, because $X_i(h)$ is non-empty and compact and U_i is continuous (since π_i, κ_{ij}, and λ_{iji} are all continuous). $\beta_{i,h}$ is furthermore convex-valued, since $X_i(h)$ is convex and U_i is quasi-concave (in fact linear) in i's own choice. Hence, $\beta_{i,h}$ is non-empty, closed-values, upper hemi-continuous, and convex-valued. These properties extend to β and γ. It follows by Katutani's fixed point theorem that γ admits a fix point. $\square$

Appendix B

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