ANALYSIS QUALIFYING EXAM FALL 2016: SOLUTIONS

Problem 1. Let m be Lebesgue measure on $\mathbb{R}$. For a subset $E \subset \mathbb{R}$ and $r \in (0, \infty)$, define

$$E_r = \{x \in \mathbb{R} \colon \operatorname{dist}(x, E) < r\}.$$

Let $E \subset \mathbb{R}$ be compact. Prove that

$$m(E) = \lim_{n \to \infty} m(E_{1/n}).$$

Solution. Recall that if (X, μ) is a measure space, and $(F_n)_{n \in \mathbb{Z}_{>0}}$ is a sequence of measurable subsets of X, and

$$F_1 \supset F_2 \supset F_3 \supset \cdots \qquad \text{and} \qquad \mu(F_1) < \infty,$$

then

$$\mu\left(\bigcap_{n=1}^{\infty} F_n \right) = \lim_{n \to \infty} m(F_n).$$

Choose $M \in [0, \infty)$ such that $E \subset [-M, M]$. Then

$$E_1 \supset E_{1/2} \supset E_{1/3} \supset E_{1/4} \supset \cdots \qquad \text{and} \qquad E_1 \subset [-M-1,\, M+1].$$

Since E is closed, we have $E = \bigcap_{n=1}^{\infty} E_{1/n}$. Since $m(E_1) \leq 2M + 2 < \infty$, it follows that

$$m\left(\bigcap_{n=1}^{\infty} E_{1/n} \right) = \lim_{n \to \infty} m(E_{1/n}).$$

Therefore

$$m(E) = \lim_{n \to \infty} m(E_{1/n}).$$

□

Alternate solution. Substitute the Dominated Convergence Theorem, used on the characteristic functions $\chi_{E_{1/n}}$, for the measure theoretic theorem stated at the beginning of the first solution. □

Problem 2. Show that, for any $n \geq 2$, the function

$$\frac{1}{(1 + x/n)^n x^{1/n}}$$

is integrable on $[1, \infty)$, and compute, with justification,

$$\lim_{n \to \infty} \int_1^{\infty} \frac{1}{(1 + x/n)^n x^{1/n}} \, dx.$$

Solution. For $n \in \mathbb{Z}_{>0}$ and $x \in [1, \infty)$, define

$$f_n(x) = \frac{1}{(1 + x/n)^n x^{1/n}}.$$

For $n \geq 2$ we have $n - 1 \geq \frac{n}{2}$, so

$$\begin{aligned} \left(1 + \frac{x}{n}\right)^n &\geq 1 + n\left(\frac{x}{n}\right) + \left(\frac{n(n-1)}{2}\right)\left(\frac{x}{n}\right)^2 \\ &\geq 1 + x + \left(\frac{n^2}{4}\right)\left(\frac{x}{n}\right)^2 = 1 + x + \frac{x^2}{4} > \frac{x^2}{4}. \end{aligned}$$

Therefore

$$0 \leq f_n(x) \leq \frac{1}{(1 + x/n)^n} \leq \frac{4}{x^2}.$$

The function $g(x) = 4/x^2$ is integrable on $[1, \infty)$, so f_n is integrable on $[0, \infty)$.

Since $\lim_{n\to\infty}(1 + x/n)^n = e^x$ and $\lim_{n\to\infty} x^{1/n} = 1$ for all $x \in [1, \infty)$, we can apply the Dominated Convergence Theorem with the dominating function g to get the first step in the following calculation:

$$\lim_{n\to\infty} \int_1^\infty \frac{1}{(1 + x/n)^n x^{1/n}}\, dx = \int_1^\infty e^{-x}\, dx = \frac{1}{e}.$$

This completes the solution. □

Problem 3. Let (X, μ) be a finite measure space. Let $(f_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence of integrable functions on X. Suppose that there is an integrable function f on X such that $f_n(x) \to f(x)$ for almost every $x \in X$. Prove that, for every $\varepsilon > 0$, there are $M \in [0, \infty)$ and a measurable subset $E \subset X$ such that $\mu(E) < \varepsilon$ and such that for all $n \in \mathbb{Z}_{>0}$ and $x \in X \setminus E$ we have $|f_n(x)| \leq M$.

Solution. We first observe that for any integrable function $g\colon X \to \mathbb{C}$ and any $r > 0$, we have

$$\mu\big(\{x \in X : |g(x)| \geq r\}\big) \leq \frac{1}{r}\|g\|_1. \tag{1}$$

Indeed, if we set

$$E = \{x \in X : |g(x)| \geq r\},$$

then $|g| \geq r\chi_E$, so

$$\frac{1}{r}\int_X |g\, d\mu \geq \frac{1}{r}\int_{\mathbb{R}} r\chi_E\, d\mu = \mu(E),$$

as desired.

Now let $\varepsilon > 0$.

Set

$$R = \frac{3\|f\|_1 + 1}{\varepsilon} \qquad \text{and} \qquad G = \{x \in X : |f(x)| \geq R\}.$$

Apply (1) with $g = f$ and $r = R$, getting $\mu(G) \leq \frac{\varepsilon}{3}$.

Next, use Egoroff's Theorem to find a measurable set $F \subset X$ such that $\mu(F) < \frac{\varepsilon}{3}$ and $f_n|_{X\setminus F} \to f|_{X\setminus F}$ uniformly. In particular, there is $N \in \mathbb{Z}_{>0}$ such that for all $x \in X \setminus F$ and all $n \geq N$, we have $|f_n(x) - f(x)| < 1$. It follows that for all $x \in X \setminus (F \cup G)$ and all $n \geq N$, we have $|f_n(x)| < R + 1$.

Now, for $n = 1, 2, \ldots, N - 1$ set

$$M_n = \frac{3N\|f_n\|_1 + 1}{\varepsilon} \qquad \text{and} \qquad E_n = \{x \in X : |f_n(x)| \geq M_n\}.$$

Apply (1) with $g = f_n$ and $r = M_n$, getting $\mu(E_n) \leq \frac{\varepsilon}{3N}$. Define

$$E = F \cup G \cup \bigcup_{n=1}^{N-1} E_n \qquad \text{and} \qquad M = \max(R+1, M_1, M_2, \dots, M_{N-1}).$$

Then

$$\mu(E) \leq \mu(F) + \mu(G) + \sum_{n=1}^{N-1} \mu(E_n) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + (N-1)\left(\frac{\varepsilon}{3N}\right) < \varepsilon.$$

Moreover, if $x \in X \setminus E$, then for $n \geq N$ we have $|f_n(x)| < R+1$ because $x \notin F \cup G$, and

$$|f_1(x)| \leq M_1, \qquad |f_2(x)| \leq M_2, \qquad \dots, \qquad |f_{N-1}(x)| \leq M_{N-1},$$

since $x \notin \bigcup_{n=1}^{N-1} E_n$. So $|f_n(x)| \leq M$ for all $n \in \mathbb{Z}_{>0}$. □

Problem 4. For any compact interval $[a, b] \subset \mathbb{R}$ and any real valued function f on $[a, b]$, let $V_{[a,b]}(f)$ denote the total variation of f on $[a, b]$.

Let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of real valued functions on $[0, 1]$. Assume that

$$\sum_{n=1}^{\infty} |f_n(0)| < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} V_{[0,1]}(f_n) < \infty.$$

Show that

$$\sum_{n=1}^{\infty} |f_n(t)| < \infty$$

for all $t \in [0, 1]$. Further, define $f(t) = \sum_{n=1}^{\infty} f_n(t)$, and prove that

$$V_{[0,1]}(f) \leq \sum_{n=1}^{\infty} V_{[0,1]}(f_n).$$

Solution. For any $t \in [0, 1]$, we have

$$\sum_{n=1}^{\infty} |f_n(t)| \leq \sum_{n=1}^{\infty} |f_n(0)| + \sum_{n=1}^{\infty} |f_n(t) - f_n(0)| \leq \sum_{n=1}^{\infty} |f_n(0)| + \sum_{n=1}^{\infty} V_{[0,1]}(f_n) < \infty.$$

Now, for any partition $\mathcal{P} = \{t_0, t_1, \dots, t_n\}$ with

$$0 = t_0 < t_1 < \cdots < t_m = 1,$$

we have

$$\begin{aligned} \sum_{j=1}^{m} |f(t_j) - f(t_{j-1})| &= \sum_{j=1}^{m} \left| \sum_{n=1}^{\infty} f_n(t_j) - \sum_{n=1}^{\infty} f_n(t_{j-1}) \right| \\ &= \sum_{j=1}^{m} \left| \sum_{n=1}^{\infty} \big(f_n(t_j) - f_n(t_{j-1}) \big) \right| \\ &\leq \sum_{j=1}^{m} \sum_{n=1}^{\infty} |f_n(t_j) - f_n(t_{j-1})| \leq \sum_{n=1}^{\infty} V_{[0,1]}(f_n). \end{aligned}$$

By taking the supremum over all such partitions $\mathcal{P}$, we get

$$V_{[0,1]}(f) \leq \sum_{n=1}^{\infty} V_{[0,1]}(f_n)$$

This completes the proof. □

Problem 5. The space l^2 is defined by

$$\Big\{(x_n)_{n\in\mathbb{Z}_{>0}}\colon x_n\in\mathbb{C} \text{ for all } n\in\mathbb{Z}_{>0} \text{ and } \textstyle\sum_{n=1}^{\infty}|x_n|^2<\infty\Big\}.$$

This space is a complex normed vector space with the norm

$$\|(x_n)_{n\in\mathbb{Z}_{>0}}\| = \left(\sum_{n=1}^{\infty}|x_n|^2\right)^{1/2}.$$

(You may use this fact without proof.)

Prove that the closed unit ball of l^2 is not compact.

Solution. For $n\in\mathbb{Z}_{>0}$, define $e_n\in l^2$ by

$$(e_n)_k = \begin{cases} 1 & k=n \\ 0 & k\neq n. \end{cases}$$

Clearly if $m,n\in\mathbb{Z}_{>0}$ are distinct, then $\|e_m-e_n\|^2=2$, so $\|e_m-e_n\|=\sqrt{2}$. Therefore the sequence $(e_n)_{n\in\mathbb{Z}_{>0}}$ has no convergent subsequence. □

Problem 6. Let E be a complex Banach space. Suppose that $\xi_1,\xi_2,\ldots,\xi_n\in E$ are linearly independent, and that $\eta_1,\eta_2,\ldots,\eta_n\in E$ are n elements. Prove that there is a bounded linear map $T\colon E\to E$ such that $T\xi_k=\eta_k$ for $k=1,2,\ldots,n$.

Solution. For $k=1,2,\ldots,n$, define the subspace $Z_k\subset E$ by

$$Z_k=\operatorname{span}\big(\{\xi_j\colon j\in\{1,2,\ldots,n\}\setminus\{k\}\}\big).$$

Then $\xi_k\notin Z_k$ because $\xi_1,\xi_2,\ldots,\xi_n$ are linearly independent. Since Z_k is finite dimensional, it is closed, so the Hahn-Banach Theorem provides a bounded linear functional $\omega_k\colon E\to\mathbb{C}$ such that $\omega_k(\xi_k)=1$ and $\omega|_{Z_k}=0$. Now define $T\in L(E)$ by

$$T\xi=\sum_{k=1}^{n}\omega_k(\xi)\eta_k$$

for $\xi\in E$. Clearly $T\xi_k=\eta_k$ for $k=1,2,\ldots,n$. Also, T is bounded because for $\xi\in E$ we have

$$\|T\xi\|\leq\sum_{k=1}^{n}\|\omega_k\|\|\eta_k\|\|\xi\|.$$

This completes the solution. □

Problem 7. Let $\varepsilon>0$, and let $f\colon B_{1+\varepsilon}(0)\to\mathbb{C}$ be a holomorphic function such that $|f(z)|=1$ for all $z\in\mathbb{C}$ with $|z|=1$, and such that $f(z)\neq 0$ for all $z\in B_1(0)$. Prove that f is constant.

Solution. The Maximum Modulus Theorem implies that $|f(z)|\leq 1$ for all $z\in\overline{B_1(0)}$. Also, $f(z)\neq 0$ for all $z\in\overline{B_1(0)}$, so there is an open set $U\subset B_{1+\varepsilon}(0)$ such that $f(z)\neq 0$ for all $z\in U$. The function $g(z)=f(z)^{-1}$ satisfies $|g(z)|=1$ for all $z\in\mathbb{C}$ such that $|z|=1$, so the Maximum Modulus Theorem implies that $|g(z)|\leq 1$ for all $z\in\overline{B_1(0)}$. Therefore $|f(z)|\geq 1$ for all $z\in\overline{B_1(0)}$. So $|f|$ has a local maximum at every point of $B_1(0)$. Thus f is constant by the condition for equality in the Maximum Modulus Theorem. □

Problem 8. Let $\Omega \subset \mathbb{C}$ be a nonempty open set, and let $f\colon [0,1] \times \Omega \to \mathbb{C}$ be a continuous function. For $t \in [0,1]$ define $f_t\colon \Omega \to \mathbb{C}$ by $f_t(z) = f(t,z)$ for $z \in \Omega$. Suppose that f_t is holomorphic for every $t \in (0,1]$. Prove that f_0 is holomorphic.

Solution. We use Morera's Theorem. So let Δ be a triangle in Ω. Let B be the boundary of Δ as a subset of $\mathbb{C}$ (rather than as a path). Then $[0,1] \times B$ is compact, and f is continuous on this set. Therefore there is $M \in [0,\infty)$ such that $|f(t,z)| \leq M$ for all $t \in [0,1]$ and $z \in B$. Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise linear parametrization of the boundary path $\partial\Delta$ of Δ. Then $L = \sup_{s\in[a,b]} |\gamma'(s)|$ is finite.

For $n \in \mathbb{Z}_{>0}$ define $h_n\colon [a,b] \to \mathbb{C}$ by $h_n(s) = f(1/n, \gamma(s))\gamma'(s)$. Also define $h\colon [a,b] \to \mathbb{C}$ by $h(s) = f(0, \gamma(s))\gamma'(s)$. Then $|h_n(s)| \leq LM$ for all $n \in \mathbb{Z}_{>0}$ and $s \in [a,b]$. Also, since f is continuous, we have $\lim_{n\to\infty} h_n(s) = h(s)$ for all $s \in [a,b]$ except for the finite set at which $\gamma'(s)$ does not exist. Using the Dominated Convergence Theorem at the second step, with the dominating function being the constant LM, we get

$$\int_{\partial\Delta} f_0(z)\,dz = \int_a^b h(s)\,ds = \lim_{n\to\infty}\int_a^b h_n(s)\,ds = \lim_{n\to\infty}\int_{\partial\Delta} f_{1/n}(z)\,dz = \lim_{n\to\infty} 0 = 0.$$

Since Δ is arbitrary and f_0 is continuous, Morera's Theorem implies that f_0 is holomorphic. □

Alternate solution. For $z_0 \in \mathbb{C}$ and $r > 0$, we let $B_r(z_0)$ be the open ball

$$B_r(z_0) = \big\{ z \in \mathbb{C}\colon |z - z_0| < r \big\}.$$

We claim that $f_{1/n} \to f_0$ uniformly on compact subsets of Ω. To prove this, let $K \subset \Omega$ be compact and let $\varepsilon > 0$. For $z \in K$, use continuity of f at $(0,z)$ to choose $\delta(z) > 0$ such that whenever $w \in \Omega$ and $t \in [0,1]$ satisfy $|w - z| < \delta(z)$ and $|t| < \delta(z)$, then

$$w \in \Omega \qquad \text{and} \qquad |f(t,w) - f(z,0)| < \frac{\varepsilon}{2}.$$

Since K is compact, there are $z_1, z_2, \ldots, z_l \in K$ such that

$$K \subset \bigcup_{k=1}^{l} B_{\delta(z_k)}(z_k).$$

Choose $N \in \mathbb{Z}_{>0}$ such that

$$\frac{1}{N} < \min\big(\delta(z_1),\, \delta(z_2),\, \ldots, \delta(z_l)\big).$$

Now let $n \in \mathbb{Z}_{>0}$ satisfy $n \geq N$. Let $z \in K$. Choose $k \in \{1, 2, \ldots, l\}$ such that $z \in B_{\delta(z_k)}(z_k)$. Then $\frac{1}{n} < \delta(z_k)$, so

$$\begin{aligned} |f_{1/n}(z) - f_0(z)| &= |f(1/n, z) - f(0,z)| \\ &\leq |f(1/n, z) - f(0, z_k)| + |f(0, z_k) - f(0,z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This completes the proof of the claim.

Given the claim, we know that $f_{1/n}$ is holomorphic for all $n \in \mathbb{Z}_{>0}$, so f is holomorphic by Theorem 10.28 of Rudin's book. □

Problem 9. Prove that there is no entire function f such that $f\big(\frac{1}{n^2}\big) = \frac{1}{n}$ for all $n \in \mathbb{Z}_{>0}$.

Solution. We argue by contradiction. Suppose there is such a function f. Then $f(0) = 0$ by continuity. Therefore there is an entire function g such that $f(z) = zg(z)$ for all $z \in \mathbb{C}$. The function g is continuous at 0. The formula $f(z) = zg(z)$ implies that $g\left(\frac{1}{n^2}\right) = n$ for $n \in \mathbb{Z}_{>0}$, which implies that $\lim_{z \to 0} g(z)$ does not exist. This contradiction shows that there is no such function f. $\square$