ALGEBRA QUALIFYING EXAM solutions — Fall 2016

I. True/False questions (9 points each). Give brief but to the point justification.

1. For $n \geq 5$ and a prime $p \leq n$, the group A_n has at least n p-Sylow subgroups.

   True. The number r of such subgroups is non-zero (because $p|(n!/2) = |A_n|$) and A_n transitively acts on them. This gives an injective proper embedding $A_n \to S_r$ (b.c. A_n is simple and $\not\simeq S_r$), so $n!/2 < r!$, i.e. $r \geq n$.

2. If a functor $F : \mathsf{C} \to \mathsf{D}$ is a left adjoint to $G : \mathsf{D} \to \mathsf{C}$ and $X = \prod_{\alpha \in A} X_\alpha$ is a product of a family of objects in C, then $F(X) \simeq \prod_{\alpha \in A} F(X_\alpha)$ in D.

   False. The free group functor F : Sets $\to$ Groups is left adjoint to the forgetful functor Groups $\to$ Sets, however we have $F_1 \simeq \mathbb{Z} \not\simeq \mathbb{Z}^2 \simeq F_1 \times F_1$.

3. If every left R-module is projective, then R is a simple ring.

   False. This is true for any semisimple ring as well, e.g. for $R = \mathbb{C} \times \mathbb{C}$.

4. If A and B are noncommutative $\mathbb{R}$-algebras of dimension 7, without nilpotents, then $A \simeq B$.

   False. The algebras $A = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{H}$ and $B = \mathbb{R} \times \mathbb{C} \times \mathbb{H}$ give a counterexample (the only one).

5. The path algebra of the A_3 quiver $\bullet \longrightarrow \bullet \longrightarrow \bullet$ is semisimple.

   False. The representation $k \xrightarrow{\mathrm{Id}} k \to 0$ of this quiver is indecomposable, but not irreducible ($0 \to k \to 0$ is a non-trivial subrepresentation). Therefore the corresponding module over the path algebra is not semisimple.

   Another way to prove non-semisimplicity is to observe that the subspace $I = \langle p \rangle$ generated by the path $p = \bullet \longrightarrow \bullet \longrightarrow \bullet$ is a nilpotent ideal.

II. Longer problems (12 points each). Do any *four* of the following problems.

1. Prove that a finite group G is nilpotent if and only if any two elements of G of relatively prime orders commute.

   For each prime $p||G|$, fix a p-Sylow subgroup $S_p \subset G$. If elements of relatively prime order commute, then elements of S_p and S_q for $p \neq q$ commute with each other and so the subgroup $H \subset G$ generated by all S_p is isomorphic to their direct product $\prod_p S_p$. But $|H| = \prod_p |S_p| = |G|$, so $G = H$ and thus G is nilpotent as a product of finite p-groups.

   Conversely, if G is nilpotent, then all its Sylow subgroups are normal, and thus pairwise commute and $G = \prod_p S_p$. Therefore the order of $g = \prod_p g_p$, where $g_p \in S_p$, is equal to the product of orders of g_p. So, if the order of $h = \prod_p h_p$ is relatively prime with the order of g, for every p at least one of the factors g_p and h_p must be equal to 1. This implies that g and h commute.

2. Let V be a $\mathbb{Q}[x]$-module corresponding to a matrix $A \in \mathrm{Mat}_n(\mathbb{Q})$. Prove that V is cyclic if and only if the ideal $\mathrm{Ann}(V)$ is generated by the characteristic polynomial of A.

   By the structure theorem of f.g. modules over PIDs, we have $V \simeq \oplus_{i=1}^r \mathbb{Q}[x]/(f_i)$, with $f_1|f_2 \ldots |f_r$; in particular $\mathrm{Ann}(V) = (f_r)$. The submodule $\mathbb{Q}[x]/(f_i)$ corresponds to the matrix A_i (the companion matrix of f_i) with the characteristic polynomial $\chi_{A_i} = f_i$, so $\chi_A = \prod_{i=1}^r f_i$. Therefore the condition $\mathrm{Ann}(V) = (\chi_A)$ is equivalent to $\deg(f_i) = 0$ for $i \neq r$ i.e. to $V \simeq \mathbb{Q}[x]/(f_r)$, i.e. to the cyclicity of V.

3. Prove that a finite-dimensional algebra over a field has finitely many simple left modules up to isomorphism.

   Let A be a finite-dimensional k-algebra with the Jacobson radical $J(A)$ and let $B = A/J(A)$. Since $J(A) = \bigcup_{V-\text{simple}} \mathrm{Ann}(V)$, the functor B-Mod $\to$ A-Mod gives a bijection (equivalence of categories) on simple modules and their isomorphism classes. Since the algebra B is finite-dimensional, and thus artinian, and $J(B) = 0$, it is semisimple. Therefore B and so A, has finitely many simple modules up to isomorphism.

4. Let $k \subset F$ be a non-trivial finite field extension. Prove that $F \otimes_k F$ is not a domain.

   Take $\alpha \in F - k$ and consider the subfield $L = k(\alpha) \subset F$. The extension $k \subset F$ is finite, so α is algebraic over k and $L = k[x]/(f)$ for an irreducible $f \in k[x]$ (minimal polynomial of α) of $\deg f \geq 2$. Since all k-modules are free and thus flat, the ring $L \otimes_k L$ is a subring of $F \otimes_k F$ and it is enough to show that $L \otimes_k L$ is not a domain. The polynomial f is no longer irreducible over L because $\alpha \in L$, i.e. $f = (x - \alpha)g$ for $g \in L[x]$. Therefore the ring $L \otimes_k L = L \otimes_k (k[x]/(f)) \simeq L[x]/(f) = L[x]/((x - \alpha)g)$ has zero divisors.

5. Prove that for any ring R, the left R-module $\mathrm{Hom}_{\mathbb{Z}}(R_R, \mathbb{Q})$ is injective.

   To prove that $V := \mathrm{Hom}_{\mathbb{Z}}(R_R, \mathbb{Q}) \in R$-Mod is injective it is enough to show that the (contravariant) functor $F = \mathrm{Hom}_R(\_\_, V) : R$-Mod $\to$ Ab is exact. Indeed, by the $\bigotimes - \mathrm{Hom}$ adjunction and the functorial isomorphism $R_R \otimes W \simeq W$, we have isomorphisms of functors $F = \mathrm{Hom}_R(\_\_, \mathrm{Hom}_{\mathbb{Z}}(R_R, \mathbb{Q})) \simeq \mathrm{Hom}_{\mathbb{Z}}(R_R \otimes_R \_\_, \mathbb{Q})) \simeq \mathrm{Hom}_{\mathbb{Z}}(\_\_, \mathbb{Q})$. Since $\mathbb{Q}$ is an injective $\mathbb{Z}$-module, the functor $\mathrm{Hom}_{\mathbb{Z}}(\_\_, \mathbb{Q})$ is exact, i.e. F is exact and so V is an injective R-module.