ALGEBRA QUALIFYING EXAM **Fall 2016**

I. True/False questions (9 points each). Give brief but to the point justification.

1. For $n \geq 5$ and a prime $p \leq n$, the group A_n has at least n p-Sylow subgroups.
2. If a functor $F : \mathsf{C} \to \mathsf{D}$ is a left adjoint to $G : \mathsf{D} \to \mathsf{C}$ and $X = \prod_{\alpha \in A} X_\alpha$ is a product of a family of objects in C, then $F(X) \simeq \prod_{\alpha \in A} F(X_\alpha)$ in D.
3. If every left R-module is projective, then R is a simple ring.
4. If A and B are noncommutative $\mathbb{R}$-algebras of dimension 7, without nilpotents, then $A \simeq B$.
5. The path algebra of the A_3 quiver $\bullet \longrightarrow \bullet \longrightarrow \bullet$ is semisimple.

II. Longer problems (12 points each). Do any *four* of the following problems.

1. Prove that a finite group G is nilpotent if and only if any two elements of G of relatively prime orders commute.
2. Let V be a $\mathbb{Q}[x]$-module corresponding to a matrix $A \in \mathrm{Mat}_n(\mathbb{Q})$. Prove that V is cyclic if and only if the ideal $\mathrm{Ann}(V)$ is generated by the characteristic polynomial of A.
3. Prove that a finite-dimensional algebra over a field has finitely many simple modules up to isomorphism.
4. Let $k \subset F$ be a non-trivial finite field extension. Prove that $F \otimes_k F$ is not a domain.
5. Prove that for any ring R, the left R-module $\mathrm{Hom}_{\mathbb{Z}}(R_R, \mathbb{Q})$ is injective.