I. True/False questions (9 points each). Give brief but to the point justification.

1. For $n \geq 5$ and a prime $p \leq n$, the group $A_n$ has at least $n$ $p$-Sylow subgroups.

2. If a functor $F : C \rightarrow D$ is a left adjoint to $G : D \rightarrow C$ and $X = \prod_{\alpha \in A} X_\alpha$ is a product of a family of objects in $C$, then $F(X) \cong \prod_{\alpha \in A} F(X_\alpha)$ in $D$.

3. If every left $R$-module is projective, then $R$ is a simple ring.

4. If $A$ and $B$ are noncommutative $\mathbb{R}$-algebras of dimension 7, without nilpotents, then $A \cong B$.

5. The path algebra of the $A_3$ quiver $\bullet \rightarrow \bullet \rightarrow \bullet$ is semisimple.

II. Longer problems (12 points each). Do any four of the following problems.

1. Prove that a finite group $G$ is nilpotent if and only if any two elements of $G$ of relatively prime orders commute.

2. Let $V$ be a $\mathbb{Q}[x]$-module corresponding to a matrix $A \in \text{Mat}_n(\mathbb{Q})$. Prove that $V$ is cyclic if and only if the ideal $\text{Ann}(V)$ is generated by the characteristic polynomial of $A$.

3. Prove that a finite-dimensional algebra over a field has finitely many simple modules up to isomorphism.

4. Let $k \subset F$ be a non-trivial finite field extension. Prove that $F \otimes_k F$ is not a domain.

5. Prove that for any ring $R$, the left $R$-module $\text{Hom}_{\mathbb{Z}}(R_R, \mathbb{Q})$ is injective.