NAME:

Topology Qualifying Exam, Winter 2017

1. Let X be a connected manifold of dimension n and $p \in X$ any point. Show that the map

$$H_i(X \smallsetminus \{p\}; \mathbb{Z}) \to H_i(X; \mathbb{Z})$$

is an isomorphism for all $i < n - 1$.

2. Let K be the Klein bottle. Compute $\pi_1(K)$ and classify all connected 2-sheeted covers of K. For each one, describe the total space.

(Hint: The total space is always homeomorphic to either K or T; you may use this statement without proof.)

3. Compute $\pi_i(\mathbb{C}P^n)$ for $1 \leq i \leq 2n+1$ and $H_i(\mathbb{C}P^n; \mathbb{Z})$ for all i.

4. Let $X = S^2 \times [0,1] \big/ \sim$, where $(p,0) \sim (-p,1)$. Compute $H_i(X;\mathbb{Z})$ for all i.

5. Let X be a nonempty, compact, connected, orientable n-manifold. Say what it means for a diffeomorphism $f : X \to X$ to be orientation-preserving. (Any of the many equivalent equivalent ways to formulate this property will be accepted.) Show that, if $X = \mathbb{C}P^6$, then every diffeomorphism $f : X \to X$ is orientation-preserving.

6. i) State the Leray-Hirsch theorem.

ii) Let $Fl_k(\mathbb{C}^n) := \{(F_0, \ldots, F_k) \mid \dim F_i = i \text{ and } F_i \subset F_{i+1}\}$. Consider the map from $Fl_k(\mathbb{C}^n)$ to $Fl_{k-1}(\mathbb{C}^n)$ given by forgetting F_k. Describe the fibers of this map.

iii) Compute the Euler characteristic $\chi(Fl_n(\mathbb{C}^n))$. You may assume that the map in part (ii) is a fiber bundle satisfying the hypotheses of the Leray-Hirsch theorem.

7. Let $X := \{(x, y, z) \in \mathbb{R}^3 \mid x^3 + xyz + y^2 = 1\}$.

i) Show that X is a 2-manifold.

ii) Consider the map $\pi : X \to \mathbb{R}^2$ taking (x, y, z) to (x, y). Find all points of X at which π fails to be a local diffeomorphism.

8. Let $X = \mathbb{R}^2 \setminus \{0\}$ and $Y = \mathbb{R}^3 \setminus \{0\}$.

i) Find submanifolds to represent all of the nonzero homogeneous elements of $H_*(X \times Y; \mathbb{Z}_2)$. Do the same for $H^*(X \times Y; \mathbb{Z}_2)$. Prove your assertions.

ii) Consider the cap product map $H_3(X \times Y; \mathbb{Z}_2) \otimes H^2(X \times Y; \mathbb{Z}_2) \to H_1(X \times Y; \mathbb{Z}_2)$. Is this map zero or nonzero?