NAME:

Topology Qualifying Exam, Winter 2017

1. Let X be a connected manifold of dimension n and $p \in X$ any point. Show that the map

$$H_i(X \smallsetminus \{p\}; \mathbb{Z}) \to H_i(X; \mathbb{Z})$$

is an isomorphism for all $i < n - 1$.

SOLUTION: By excision, $H_i(X, X \smallsetminus \{p\}; \mathbb{Z}) \cong H_i(D^n, D^n \smallsetminus \{0\}; \mathbb{Z})$, which is isomorphic to $\mathbb{Z}$ if $i = n$ and 0 otherwise. Thus the exact sequence

$$H_{i+1}(X, X \smallsetminus \{p\}; \mathbb{Z}) \to H_i(X \smallsetminus \{p\}; \mathbb{Z}) \to H_i(X; \mathbb{Z}) \to H_i(X, X \smallsetminus \{p\}; \mathbb{Z})$$

tells us that the middle map is an isomorphism whenever $i < n - 1$.

2. Let K be the Klein bottle. Compute $\pi_1(K)$ and classify all connected 2-sheeted covers of K. For each one, describe the total space.

(Hint: The total space is always homeomorphic to either K or T; you may use this statement without proof.)

SOLUTION: The standard picture of K is that of a CW-complex with one 0-cell, two 1-cells, and one 2-cell. That means that $\pi_1(K)$ has a presentation with two generators (which I'll call a and b) and one relation. Going around the boundary of the 2-cell, we see that the relation is $abab^{-1}$.

Connected 2-sheeted covers of K correspond to conjugacy classes of index 2 subgroups of $\pi_1(K)$. But all index 2 subgroups are normal, so they simply correspond to index 2 subgroups, which in turn correspond to surjective homomorphisms from $\pi_1(K)$ to $\mathbb{Z}_2$. There are three such: a goes to 1 and b goes to 0, a goes to 0 and b goes to 1, or a and b both go to 1. In the first case, the kernel is $\langle a^2, b\rangle \cong \pi_1(K)$, so the cover is homeomorphic to K. In the second case, the kernel is $\langle a, b^2\rangle \cong \mathbb{Z}^2 \cong \pi_1(T)$, so the cover is homeomorphic to T. In the third case, the kernel is $\langle a^2, ab\rangle \cong \pi_1(K)$, so the cover is homeomorphic to K.

3. Compute $\pi_i(\mathbb{C}P^n)$ for $1 \leq i \leq 2n+1$ and $H_i(\mathbb{C}P^n; \mathbb{Z})$ for all i.

SOLUTION: Using the long exact sequence associated with the fibration $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$, we see that $\pi_1(\mathbb{C}P^n) = 0$, $\pi_2(\mathbb{C}P^n) \cong \mathbb{Z}$, $\pi_i(\mathbb{C}P^n) = 0$ for $3 \leq i \leq 2n$, and $\pi_{2n+1}(\mathbb{C}P^n) = \mathbb{Z}$.

Using the fact that $\mathbb{C}P^n$ has a cell structure with one cell in dimension $2i$ for all $0 \leq i \leq n$, we see that $H_{2i}(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}$ for all $0 \leq i \leq n$, and all other homology groups are zero.

4. Let $X = S^2 \times [0,1] \big/ \sim$, where $(p,0) \sim (-p,1)$. Compute $H_i(X;\mathbb{Z})$ for all i.

SOLUTION: Let $A := S^2 \times (1/5, 4/5) \subset X$, and let B be the image of $S^2 \times [0, 2/5) \cup S^2 \times (3/5, 1]$ in X. We will apply Mayer-Vietoris to the cover $X = A \cup B$. The intersection $A \cap B$ has two components, $U = S^2 \times (1/5, 2/5)$ and $V = S^2 \times (3/5, 4/5)$.

The spaces A, B, U, and V are all homotopy equivalent to S^2. For A, B, and U, we fix our homotopy equivalence to be given by the inclusion of $S^2 \cong \{3/10\} \times S^2$, which lies in all three open sets. For V, we fix our homotopy equivalence by the inclusion of V into $A \simeq S^2$. This way, the inclusion of U into A, the inclusion of U into B, and the inclusion of V into A all look like the identity map on S^2. The only funny map is the inclusion of V into B, which looks like the antipodal map on S^2.

It's easy to see from the Mayer-Vietoris sequence that $H_0(X;\mathbb{Z}) = \mathbb{Z}$ and $H_i(X;\mathbb{Z}) = 0$ for all $i \geq 4$. The group $H_1(X;\mathbb{Z})$ is isomorphic to the kernel of the map

$$H_2(U) \oplus H_2(V) \to H_2(A) \oplus H_2(B).$$

This map is given by the matrix

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

(the second row gets a minus sign from the definition of the Mayer-Vietoris sequence), and its kernel is isomorphic to $\mathbb{Z}$. To compute $H_2(X;\mathbb{Z})$ and $H_3(X;\mathbb{Z})$, we observe that we have

$$0 \to H_3(X) \to H_2(U) \oplus H_2(V) \to H_2(A) \oplus H_2(B) \to H_2(X) \to 0,$$

where the map in the middle is given by the matrix

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

(The second row gets a minus sign from the definition of the Mayer-Vietoris sequence, and the bottom-right entry, which corresponds to the inclusion of V into B, gets an additional minus sign from the degree of the antipodal map on S^2.) This matrix has determinant 2, which means that it has kernel 0 and cokernel $\mathbb{Z}_2$. Thus $H_3(X;\mathbb{Z}) = 0$ and $H_2(X;\mathbb{Z}) \cong \mathbb{Z}_2$.

5. Let X be a nonempty, compact, connected, orientable n-manifold. Say what it means for a diffeomorphism $f : X \to X$ to be orientation-preserving. (Any of the many equivalent equivalent ways to formulate this property will be accepted.) Show that, if $X = \mathbb{C}P^6$, then every diffeomorphism $f : X \to X$ is orientation-preserving.

SOLUTION: The map f is orientation preserving if and only if the induced map $f^* : H^n(X;\mathbb{Z}) \to H^n(X;\mathbb{Z})$ is the identity. We have $H^*(\mathbb{C}P^6;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/\langle\alpha^7\rangle$, and $H^{12}(\mathbb{C}P^6;\mathbb{Z})$ is generated by α^6. Any diffeomorphism f must send α to $\pm\alpha$, thus it sends α^6 to itself.

6. i) State the Leray-Hirsch theorem.

SOLUTION: Let $X \to B$ be a fiber bundle, and suppose that for each fiber F, the restriction map from the cohomology of X to the cohomology of F is surjective. Assume also that $H^n(F; R)$ is a finitely generated free R-module for all n. Then there exists a (noncanonical) isomorphism $H^*(X) \cong H^*(B) \otimes H^*(F)$ of graded $H^*(B)$-modules.

ii) Let $Fl_k(\mathbb{C}^n) := \{(F_0, \ldots, F_k) \mid \dim F_i = i \text{ and } F_i \subset F_{i+1}\}$. Consider the map from $Fl_k(\mathbb{C}^n)$ to $Fl_{k-1}(\mathbb{C}^n)$ given by forgetting F_k. Describe the fibers of this map.

SOLUTION: An element of the fiber over $(F_0, \ldots, F_{k-1})$ is given by a k-dimensional subspace $F_k \subset \mathbb{C}^n$ containing F_{k-1}. This is the same as a line in $\mathbb{C}^n/F_{k-1}$, which is a vector space of dimension $n - k + 1$. This the fiber is isomorphic to $\mathbb{C}P^{n-k}$.

iii) Compute the Euler characteristic $\chi(Fl_n(\mathbb{C}^n))$. You may assume that the map in part (ii) is a fiber bundle satisfying the hypotheses of the Leray-Hirsch theorem.

SOLUTION: Applying the Leray-Hirsch theorem to the bundle in part (ii), we obtain a graded vector space isomorphism $H^*(Fl_k(\mathbb{C}^n)) \cong H^*(Fl_{k-1}(\mathbb{C}^n)) \otimes H^*(Fl_{k-1}(\mathbb{C}^n))$. Inductively, we conclude that

$$H^*(Fl_n(\mathbb{C}^n)) \cong H^*(\mathbb{C}P^1) \otimes H^*(\mathbb{C}P^2) \otimes \cdots \otimes H^*(\mathbb{C}P^{n-1}).$$

This tells us that all the cohomology is concentrated in even degree, which means that the Euler characteristic is equal to the total dimension, which is $2 \cdot 3 \cdots n = n!$.

7. Let $X := \{(x, y, z) \in \mathbb{R}^3 \mid x^3 + xyz + y^2 = 1\}$.

i) Show that X is a 2-manifold.

SOLUTION: Let $f(x, y, z) = x^3 + xyz + y^2 - 1$, so that $X = f^{-1}(0)$. We have

$$f_x(x, y, z) = 3x^2 + yz, \qquad f_y(x, y, z) = xz + 2y, \qquad \text{and} \qquad f_z(x, y, z) = xy.$$

These partial derivatives all vanish if and only if $x = y = 0$, which cannot happen on X, so X is a 2-manifold.

ii) Consider the map $\pi : X \to \mathbb{R}^2$ taking (x, y, z) to (x, y). Find all points of X at which π fails to be a local diffeomorphism.

SOLUTION: The map π fails to be a local diffeomorphism at (x, y, z) if and only if the vector $(0, 0, 1)$ lies in $T_{(x,y,z)}X$. By the computation in part (i), this holds if and only if $xy = 0$. Thus the bad points are

$$\{(0, \pm 1, z) \mid z \in \mathbb{R}\} \cup \{(1, 0, z) \mid z \in \mathbb{R}\}.$$

8. Let $X = \mathbb{R}^2 \smallsetminus \{0\}$ and $Y = \mathbb{R}^3 \smallsetminus \{0\}$.

i) Find submanifolds to represent all of the nonzero homogeneous elements of $H_*(X \times Y; \mathbb{Z}_2)$. Do the same for $H^*(X \times Y; \mathbb{Z}_2)$. Prove your assertions.

SOLUTION: By the Künneth theorem, $H^i(X \times Y; \mathbb{Z}_2) \cong \mathbb{Z}_2$ for $0 \leq i \leq 3$ and 0 otherwise. Since homology is dual to cohomology, the same is true of homology. Thus we just need to find one nonzero class in each degree less than or equal to 3, both in homology and cohomology.

Let $C \subset X$ be the circle of radius 1 and let $S \in Y$ be the sphere of radius 1. Let $R \subset X$ be the positive x-axis and $T \subset Y$ the positive x-axis. Let $p = C \cap R \in X$ and $q = S \cap T \in Y$.

Then we have $[(p, q)] \in H_0(X \times Y; \mathbb{Z}_2)$, $[C \times \{q\}] \in H_1(X \times Y; \mathbb{Z}_2)$, $[\{p\} \times S] \in H_2(X \times Y; \mathbb{Z}_2)$, and $[C \times S] \in H_3(X \times Y; \mathbb{Z}_2)$.

Let $R = \{(x, 0) \mid x > 0\} \subset X$, and let $T = \{(x, 0, 0) \mid x > 0\} \subset Y$. Then we have $(X \times Y) \in H^0(X \times Y; \mathbb{Z}_2)$, $(R \times Y) \in H^1(X \times Y; \mathbb{Z}_2)$, $(X \times T) \in H^2(X \times Y; \mathbb{Z}_2)$, and $(R \times T) \in H^3(X; \mathbb{Z}_2)$.

I claim that all of these classes are nonzero. To see this, we note that $X \times Y$ and (p, q) intersect transversely at (p, q), therefore the classes $[(p, q)]$ and $(X \times Y)$ pair nontrivially. The same is true for $C \times \{q\}$ and $X \times T$, $\{p\} \times S$ and $R \times Y$, and $C \times S$ and $R \times T$.

ii) Consider the cap product map $H_3(X \times Y; \mathbb{Z}_2) \otimes H^2(X \times Y; \mathbb{Z}_2) \to H_1(X \times Y; \mathbb{Z}_2)$. Is this map zero or nonzero?

SOLUTION: We have

$$[C \times S] \frown (X \times T) = [C \times \{q\}] \neq 0,$$

so the map is nonzero.