NAME:

Topology Qualifying Exam, Fall 2016

1. i) State the van Kampen theorem.

ii) Let X be a connected manifold of dimension $n > 2$, and $p \neq q \in X$ any two points. Show that the map

$$\pi_1(X \setminus \{p\}, q) \to \pi_1(X, q)$$

is an isomorphism.

2. For each of the following subgroups of $\mathbb{Z} * \mathbb{Z}$, draw a covering space of the figure eight with this as its fundamental group and determine whether or not the subgroup is normal.

i) $\langle a^3, b, aba^{-1}, a^{-1}ba \rangle$

ii) $\langle a^2, b^2, aba, bab \rangle$

3. i) Prove that $\pi_n(\vee^r S^n) \cong \mathbb{Z}^r$ for all $n > 1$.

ii) Let $X = \mathbb{R}P^2 \vee S^2$. Compute $\pi_i(X)$ for $i = 1, 2$ and compute $H_i(X; \mathbb{Z})$ for all i.

4. i) Let X be a compact n-manifold (without boundary) with n odd. Show that the Euler characteristic $\chi(X)$ is equal to zero.

ii) Show that there does not exist a 5-manifold (with boundary) W such that $\partial W \cong \mathbb{C}P^2$.

(Hint: Let $X := W \bigcup_{\partial W} W$, which you may assume is a manifold.)

5. i) State the Künneth theorem.

ii) Show that there do not exist 3-manifolds X and Y such that $X \times Y \cong \mathbb{R}P^6$.

6. Let X be a nonempty, compact, connected n-manifold. Show that the torsion subgroup of $H_{n-1}(X;\mathbb{Z})$ is equal to 0 if X is orientable and $\mathbb{Z}_2$ if X is not orientable.

7. Let $X := \{(x, y, z) \in \mathbb{R}^3 \mid x^3 + y^3 + z^3 + 3yz = 1\}$. Show that X is an orientable 2-manifold.

8. i) State the Alexander duality theorem.

ii) Let $X := \mathbb{R}^3 \smallsetminus \{(x, y, 0) \mid x^2 + y^2 = 1\}$ be the complement of an unknotted circle in $\mathbb{R}^3$. Compute $H_*(X; \mathbb{Z}_2)$ and $H^*(X; \mathbb{Z}_2)$. (Hint: Start by writing X as the complement of a subset of S^3.)

iii) Find submanifolds to represent all of the nonzero homogeneous elements of $H_*(X; \mathbb{Z}_2)$. Do the same for $H^*(X; \mathbb{Z}_2)$. Prove your assertions.

iv) Consider the cap product map $H_2(X; \mathbb{Z}_2) \otimes H^1(X; \mathbb{Z}_2) \to H_1(X; \mathbb{Z}_2)$. Is this map zero or nonzero?