NAME:

Topology Qualifying Exam, Fall 2016

1. i) State the van Kampen theorem.

SOLUTION: Let $X = U \cup V$ be an open cover, with U, V, and $U \cap V$, all path connected. Let $i : U \to X$ and $j : V \to X$ be the inclusions. Then for any $p \in U \cap V$, $\pi_1(X, p)$ is isomorphic to the quotient of $\pi_1(U, p) * \pi_1(V, p)$ by the subgroup generated by elements of the form $(i_*\alpha)(j_*\alpha)^{-1}$ for all $\alpha \in \pi_1(U \cap V, p)$.

ii) Let X be a connected manifold of dimension $n > 2$, and $p \neq q \in X$ any two points. Show that the map

$$\pi_1(X \smallsetminus \{p\}, q) \to \pi_1(X, q)$$

is an isomorphism.

SOLUTION: Let $D \subset X$ be an open n-dimensional disk that contains both p and q. We apply the van Kampen theorem to the cover $X = D \cup \Big(X \smallsetminus \{p\}\Big)$. We have $\pi_1(D, q) = 0$, and since $n > 2$, we also have $\pi_1(D \smallsetminus \{p\}, q) \cong \pi_1(S^{n-1}) = 0$. Thus the van Kampen theorem simply says that the map $\pi_1(X \smallsetminus \{p\}, q) \to \pi_1(X, q)$ is an isomorphism.

2. For each of the following subgroups of $\mathbb{Z} * \mathbb{Z}$, draw a based covering space of the figure eight with this as its fundamental group and determine whether or not the subgroup is normal.

i) $\langle a^3, b, aba^{-1}, a^{-1}ba \rangle$

SOLUTION: The picture should be that of a triangle (with edges labeled a and oriented cyclically) along with a loop at each vertex (labeled b). This picture has a symmetry group that acts transitively on the vertices, which means that the cover is regular and the corresponding group is a normal subgroup.

ii) $\langle a^2, b^2, aba, bab \rangle$

SOLUTION: The picture should be that of two side-by-side "kissing" figure eights, with the middle vertex as the distinguished one. This picture has no nontrivial symmetries, which means that cover is irregular and the corresponding subgroup is not normal.

3. i) Prove that $\pi_n(\vee^r S^n) \cong \mathbb{Z}^r$ for all $n > 1$.

SOLUTION: The space $\vee^r S^n$ is equal to the $(2n-1)$-skeleton of $(S^n)^r$. Since $n < 2n-1$, $\pi_n(\vee^r S^n) \cong \pi_n((S^n)^r) \cong \mathbb{Z}^r$.

ii) Let $X = \mathbb{R}P^2 \vee S^2$. Compute $\pi_i(X)$ for $i = 1, 2$ and compute $H_i(X; \mathbb{Z})$ for all i.

SOLUTION: We have $\pi_1(X) \cong \pi_1(\mathbb{R}P^2) * \pi_1(S^2) \cong \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$.

The universal cover of X is a chain of three spheres, which is homotopy equivalent to a wedge of three spheres. Thus, by part (i),

$$\pi_2(X) \cong \pi_2(S^2 \vee S^2 \vee S^2) \cong \mathbb{Z}^3.$$

Since X is connected, $H_0(X; \mathbb{Z}) \cong \mathbb{Z}$. For all $i > 0$, $H_i(X; \mathbb{Z}) \cong H_i(\mathbb{R}P^2; \mathbb{Z}) \oplus H_i(S^2; \mathbb{Z})$. Thus $H_1(X; \mathbb{Z}) \cong \mathbb{Z}_2$, $H_2(X; \mathbb{Z}) \cong \mathbb{Z}$, and $H_i(X; \mathbb{Z}) = 0$ for all $i > 2$.

4. i) Let X be a compact n-manifold (without boundary) with n odd. Show that the Euler characteristic $\chi(X)$ is equal to zero.

SOLUTION: We have $\chi(X) = \sum_{i=0}^{n}(-1)^i \dim H^i(X;\mathbb{Z}_2)$. By Poincaré Duality, we have $\dim H^i(X;\mathbb{Z}_2) = \dim H^{n-i}(X;\mathbb{Z}_2)$, so

$$\chi(X) = \sum_{i=0}^{n}(-1)^i \dim H^{n-i}(X;\mathbb{Z}_2) = \sum_{i=0}^{n}(-1)^{n-i} \dim H^i(X;\mathbb{Z}_2) = -\chi(X).$$

Thus $\chi(X) = 0$.

ii) Show that there does not exist a compact 5-manifold (with boundary) W such that $\partial W \cong \mathbb{C}P^2$.

(Hint: Let $X := W \bigcup_{\partial W} W$, which you may assume is a manifold.)

SOLUTION: Recall that, if we have a long exact sequence of vector spaces, the alternating sum of the dimensions of the vector spaces is equal to zero. Applying this fact to the Mayer-Vietoris sequences associated with the cover of X by two copies of W, we obtain the equation

$$\chi(X) = 2\chi(W) - \chi(\partial W).$$

But $\partial W \cong \mathbb{C}P^2$, so $\chi(\partial W) = \chi(\mathbb{C}P^2) = 3$, which means that $\chi(X)$ must be odd. This contradicts part (i).

5. i) State the Künneth theorem.

SOLUTION: Suppose that X and Y are CW-complexes with $H^k(Y;R)$ a finitely generated free R-module for all k. Then $H^*(X \times Y;R) \cong H^*(X;R) \otimes H^*(Y;R)$.

ii) Show that there do not exist 3-manifolds X and Y such that $X \times Y \cong \mathbb{R}P^6$.

SOLUTION: Suppose there did. Clearly X and Y must be compact, connected, and nonempty. Then

$$\mathbb{Z}_2 \cong H^3(\mathbb{R}P^6;\mathbb{Z}_2) \supset \Big(H^0(X;\mathbb{Z}_2) \otimes H^3(Y;\mathbb{Z}_2)\Big) \oplus \Big(H^3(X;\mathbb{Z}_2) \otimes H^0(Y;\mathbb{Z}_2)\Big) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$

which gives a contradiction.

6. Let X be a nonempty, compact, connected n-manifold. Show that the torsion subgroup of $H_{n-1}(X;\mathbb{Z})$ is equal to 0 if X is orientable and $\mathbb{Z}_2$ if X is not orientable.

SOLUTION: First suppose that X is orientable. Then Poincaré duality tells us that

$$H_{n-1}(X;\mathbb{Z}) \cong H^1(X;\mathbb{Z}),$$

and the universal coefficient theorem tells us that

$$H^1(X;\mathbb{Z}) \cong \mathrm{Hom}(H_1(X;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}(H_0(X;\mathbb{Z}),\mathbb{Z}).$$

Since $H_0(X;\mathbb{Z})$ is free, the second summand vanishes. Since $\mathbb{Z}$ is torsion-free, so is the first summand.

Now suppose that X is not orientable, and let p be any prime. The universal coefficient theorem tells us that

$$H^n(X;\mathbb{Z}_p) \cong \mathrm{Hom}(H_n(X;\mathbb{Z}),\mathbb{Z}_p) \oplus \mathrm{Ext}(H_{n-1}(X;\mathbb{Z}),\mathbb{Z}_p).$$

Since X is not orientable, $H_n(X;\mathbb{Z}) = 0$, so the first summand vanishes. The second summand is equal to the p-torsion part of $H_{n-1}(X;\mathbb{Z})$. If p is odd, then the left-hand side is zero, so the p-torsion vanishes. If $p = 2$, then the left-hand side is $\mathbb{Z}_2$, so the 2-torsion is $\mathbb{Z}_2$.

7. Let $X := \{(x, y, z) \in \mathbb{R}^3 \mid x^3 + y^3 + z^3 + 3yz = 2\}$. Show that X is an orientable 2-manifold.

SOLUTION: Let $f(x, y, z) = x^3 + y^3 + z^3 + 3yz - 2$, so that $X = f^{-1}(0)$. We have

$$f_x(x, y, z) = 3x^2, \qquad f_y(x, y, z) = 3y^2 + 3z, \qquad \text{and} \qquad f_z(x, y, z) = 3z^2 + 3y.$$

These partial derivatives all vanish exactly at the points $(0, 0, 0)$ and $(0, -1, -1)$. Since neither of these points lies in X, 0 is a regular value of f, so X is a 2-manifold.

Since $\mathbb{R}^3$ is orientable, X is orientable if and only if it is coorientable as a submanifold of $\mathbb{R}^3$. That is, we need to show that the normal bundle of X in $\mathbb{R}^3$ is trivial. But the normal bundle of X in $\mathbb{R}^3$ is canonically trivialized by the isomorphism

$$df_{(x,y,z)} : \mathbb{R}^3 / T_{(x,y,z)} X \to \mathbb{R}.$$

8. i) State the Alexander duality theorem.

SOLUTION: If K is a closed submanifold of S^n, then $\tilde{H}_i(S^n \smallsetminus K; R) \cong \tilde{H}^{n-i-1}(K; R)$.

ii) Let $X := \mathbb{R}^3 \smallsetminus \{(x, y, 0) \mid x^2 + y^2 = 1\}$ be the complement of an unknotted circle in $\mathbb{R}^3$. Compute $H_*(X; \mathbb{Z}_2)$ and $H^*(X; \mathbb{Z}_2)$. (Hint: Start by writing X as the complement of a subset of S^3.)

SOLUTION: Applying Alexander duality to $K = S^1 \sqcup \{\infty\} \subset S^3$, we see that $H^0(X; \mathbb{Z}_2) = H^1(X; \mathbb{Z}_2) = H^2(X; \mathbb{Z}_2) = \mathbb{Z}_2$, with all other cohomology groups being zero. Since homology groups are dual to cohomology groups, the same is true in homology.

iii) Find submanifolds to represent all of the nonzero homogeneous elements of $H_*(X; \mathbb{Z}_2)$. Do the same for $H^*(X; \mathbb{Z}_2)$. Prove your assertions.

SOLUTION: Let $p \in X$ be any point, let $C \in X$ be a circle that links once with the knot, and let $S \in X$ be a sphere of raduis 2 surrounding the knot. Then we have $[p] \in H_0(X; \mathbb{Z}_2)$, $[C] \in H_1(X; \mathbb{Z}_2)$, and $[S] \in H_2(X; \mathbb{Z}_2)$.

Let $D = \{(x, y, 0) \mid x^2 + y^2 < 1\} \subset X$, and let $R = \{(x, 0, 0) \mid x > 1\}$. Then we have $(X) \in H^0(X; \mathbb{Z}_2)$, $(D) \in H^1(X; \mathbb{Z}_2)$, and $(R) \in H^2(X; \mathbb{Z}_2)$.

I claim that all of these classes are nonzero. To see this, we note that X and p intersect transversely at p, therefore the classes $[p]$ and (X) pair nontrivially. Similarly, C and D intersect transversely at a single point, and S and R intersect transversely at a single point.

iv) Consider the cap product map $H_2(X; \mathbb{Z}_2) \otimes H^1(X; \mathbb{Z}_2) \to H_1(X; \mathbb{Z}_2)$. Is this map zero or nonzero?

We have $[S] \frown (D) = [S \cap D] = [\emptyset] = 0$, hence the map is zero.