ANALYSIS QUALIFYING EXAM FALL 2016

Problem 1. Let m be Lebesgue measure on $\mathbb{R}$. For a subset $E \subset \mathbb{R}$ and $r \in (0, \infty)$, define

$$E_r = \{x \in \mathbb{R} \colon \operatorname{dist}(x, E) < r\}.$$

Let $E \subset \mathbb{R}$ be compact. Prove that

$$m(E) = \lim_{n \to \infty} m(E_{1/n}).$$

Problem 2. Show that, for any $n \geq 2$, the function

$$\frac{1}{(1 + x/n)^n x^{1/n}}$$

is integrable on $[1, \infty)$, and compute, with justification,

$$\lim_{n \to \infty} \int_1^\infty \frac{1}{(1 + x/n)^n x^{1/n}} \, dx.$$

Problem 3. Let (X, μ) be a finite measure space. Let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of integrable functions on X. Suppose that there is an integrable function f on X such that $f_n(x) \to f(x)$ for almost every $x \in X$. Prove that, for every $\varepsilon > 0$, there are $M \in [0, \infty)$ and a measurable subset $E \subset X$ such that $\mu(E) < \varepsilon$ and such that for all $n \in \mathbb{Z}_{>0}$ and $x \in X \setminus E$ we have $|f_n(x)| \leq M$.

Problem 4. For any compact interval $[a, b] \subset \mathbb{R}$ and any real valued function f on $[a, b]$, let $V_{[a,b]}(f)$ denote the total variation of f on $[a, b]$.

Let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of real valued functions on $[0, 1]$. Assume that

$$\sum_{n=1}^\infty |f_n(0)| < \infty \qquad \text{and} \qquad \sum_{n=1}^\infty V_{[0,1]}(f_n) < \infty.$$

Show that

$$\sum_{n=1}^\infty |f_n(t)| < \infty$$

for all $t \in [0, 1]$. Further, define $f(t) = \sum_{n=1}^\infty f_n(t)$, and prove that

$$V_{[0,1]}(f) \leq \sum_{n=1}^\infty V_{[0,1]}(f_n).$$

Problem 5. The space l^2 is defined by

$$\Big\{ (x_n)_{n \in \mathbb{Z}_{>0}} \colon x_n \in \mathbb{C} \text{ for all } n \in \mathbb{Z}_{>0} \text{ and } \textstyle\sum_{n=1}^\infty |x_n|^2 < \infty \Big\}.$$

This space is a complex normed vector space with the norm

$$\|(x_n)_{n \in \mathbb{Z}_{>0}}\| = \left(\sum_{n=1}^\infty |x_n|^2 \right)^{1/2}.$$

(You may use this fact without proof.)

Prove that the closed unit ball of l^2 is not compact.

Problem 6. Let E be a complex Banach space. Suppose that $\xi_1, \xi_2, \dots, \xi_n \in E$ are linearly independent, and that $\eta_1, \eta_2, \dots, \eta_n \in E$ are n elements. Prove that there is a bounded linear map $T\colon E \to E$ such that $T\xi_k = \eta_k$ for $k = 1, 2, \dots, n$.

Problem 7. Let $\varepsilon > 0$, and let $f\colon B_{1+\varepsilon}(0) \to \mathbb{C}$ be a holomorphic function such that $|f(z)| = 1$ for all $z \in \mathbb{C}$ with $|z| = 1$, and such that $f(z) \neq 0$ for all $z \in B_1(0)$. Prove that f is constant.

Problem 8. Let $\Omega \subset \mathbb{C}$ be a nonempty open set, and let $f\colon [0,1] \times \Omega \to \mathbb{C}$ be a continuous function. For $t \in [0,1]$ define $f_t\colon \Omega \to \mathbb{C}$ by $f_t(z) = f(t,z)$ for $z \in \Omega$. Suppose that f_t is holomorphic for every $t \in (0,1]$. Prove that f_0 is holomorphic.

Problem 9. Prove that there is no entire function f such that $f\left(\frac{1}{n^2}\right) = \frac{1}{n}$ for all $n \in \mathbb{Z}_{>0}$.