**Problem 1.** Let $m$ be Lebesgue measure on $\mathbb{R}$. For a subset $E \subset \mathbb{R}$ and $r \in (0, \infty)$, define

$$E_r = \{ x \in \mathbb{R} : \text{dist}(x, E) < r \}.$$

Let $E \subset \mathbb{R}$ be compact. Prove that

$$m(E) = \lim_{n \to \infty} m(E_{1/n}).$$

**Problem 2.** Show that, for any $n \geq 2$, the function

$$\frac{1}{(1 + x/n)^{n^{1/n}}}$$

is integrable on $[1, \infty)$, and compute, with justification,

$$\lim_{n \to \infty} \int_1^{\infty} \frac{1}{(1 + x/n)^{n^{1/n}}} \, dx.$$

**Problem 3.** Let $(X, \mu)$ be a finite measure space. Let $(f_n)_{n \in \mathbb{Z}^+}$ be a sequence of integrable functions on $X$ such that $f_n(x) \to f(x)$ for almost every $x \in X$. Prove that, for every $\varepsilon > 0$, there are $M \in [0, \infty)$ and a measurable subset $E \subset X$ such that $\mu(E) < \varepsilon$ and such that for all $n \in \mathbb{Z}^+$ and $x \in X \setminus E$ we have $|f_n(x)| \leq M$.

**Problem 4.** For any compact interval $[a, b] \subset \mathbb{R}$ and any real valued function $f$ on $[a, b]$, let $V_{[a,b]}(f)$ denote the total variation of $f$ on $[a, b]$.

Let $(f_n)_{n \in \mathbb{Z}^+}$ be a sequence of real valued functions on $[0, 1]$. Assume that

$$\sum_{n=1}^{\infty} |f_n(0)| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} V_{[0,1]}(f_n) < \infty.$$

Show that

$$\sum_{n=1}^{\infty} |f_n(t)| < \infty$$

for all $t \in [0, 1]$. Further, define $f(t) = \sum_{n=1}^{\infty} f_n(t)$, and prove that

$$V_{[0,1]}(f) \leq \sum_{n=1}^{\infty} V_{[0,1]}(f_n).$$

**Problem 5.** The space $l^2$ is defined by

$$\{(x_n)_{n \in \mathbb{Z}^+} : x_n \in \mathbb{C} \text{ for all } n \in \mathbb{Z}^+ \text{ and } \sum_{n=1}^{\infty} |x_n|^2 < \infty \}.$$

This space is a complex normed vector space with the norm

$$\|(x_n)_{n \in \mathbb{Z}^+}\| = \left( \sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2}.$$

(You may use this fact without proof.)

Prove that the closed unit ball of $l^2$ is not compact.
Problem 6. Let $E$ be a complex Banach space. Suppose that $\xi_1, \xi_2, \ldots, \xi_n \in E$ are linearly independent, and that $\eta_1, \eta_2, \ldots, \eta_n \in E$ are $n$ elements. Prove that there is a bounded linear map $T: E \rightarrow E$ such that $T\xi_k = \eta_k$ for $k = 1, 2, \ldots, n$.

Problem 7. Let $\varepsilon > 0$, and let $f: B_{1+\varepsilon}(0) \rightarrow \mathbb{C}$ be a holomorphic function such that $|f(z)| = 1$ for all $z \in \mathbb{C}$ with $|z| = 1$, and such that $f(z) \neq 0$ for all $z \in B_1(0)$. Prove that $f$ is constant.

Problem 8. Let $\Omega \subset \mathbb{C}$ be a nonempty open set, and let $f: [0, 1] \times \Omega \rightarrow \mathbb{C}$ be a continuous function. For $t \in [0, 1]$ define $f_t: \Omega \rightarrow \mathbb{C}$ by $f_t(z) = f(t, z)$ for $z \in \Omega$. Suppose that $f_t$ is holomorphic for every $t \in (0, 1]$. Prove that $f_0$ is holomorphic.

Problem 9. Prove that there is no entire function $f$ such that $f\left(\frac{1}{n^2}\right) = \frac{1}{n}$ for all $n \in \mathbb{Z}_{>0}$.