TOPOLOGY QUALIFYING EXAM, UNIVERSITY OF OREGON, WINTER 2016

Each problem is worth 10 points. When proving statements, you may use results from Hatcher's and May's textbooks or from Botwinnik's lecture notes. Do not forget to sign your paper. Good luck!

1. Prove that any continuous map $f : \mathbb{R}P^2 \to S^1$ is homotopy equivalent to a constant map.

2. Compute $Tor_*^{\mathbb{Z}}(\mathbb{Z}/8, \mathbb{Z}/6)$ and $Ext^*_{\mathbb{Z}}(\mathbb{Z}/8, \mathbb{Z}/6)$.

3. (a) Calculate the degree of the map $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ given by $[z_0 : z_1] \mapsto [z_0^d : z_1^d]$.
(b) Define $g : \mathbb{C}P^2 \to \mathbb{C}P^2$ by $[z_0 : z_1 : z_2] \mapsto [z_0^d : z_1^d : z_2^d]$. Calculate the induced map $g^* : H^4(\mathbb{C}P^2, \mathbb{Z}) \to H^4(\mathbb{C}P^2, \mathbb{Z})$.

4. (a) State the Lefschetz Fixed Point Theorem.
(b) Let $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ be a map. Prove that f always has a fixed point. Give an example that the above statement fails for a map $f : \mathbb{R}P^{2n+1} \to \mathbb{R}P^{2n+1}$.

5. Let M be a compact connected oriented surface of genus 2 *i.e.*, a sphere with 2 handles.
(a) How many connected 2-fold covers $K \to M$ are there?
(b) For each such cover, find the genus of K.

6. (a) State what it means for a manifold M to be orientable.
(b) Let G be a topological group, which is also a manifold. Prove that G is orientable.

7. (a) Let X be a topological space, $U, W \subset X$ open subsets. Give a definition of the relative cup product

$$H^p(X, U, \mathbb{Z}) \otimes H^q(X, W, \mathbb{Z}) \to H^{p+q}(X, U \cup W, \mathbb{Z}).$$

(b) Show that, for any space Y and every $p, q > 0$, the cup product

$$H^p(\Sigma Y, \mathbb{Z}) \otimes H^q(\Sigma Y, \mathbb{Z}) \to H^{p+q}(\Sigma Y, \mathbb{Z})$$

is 0.

8. (a) For $a \in \pi_n(X), b \in \pi_k(X)$, define the Whitehead product $[a, b] \in \pi_{n+k-1}(X)$.
(b) Prove that, for any X, $a \in \pi_n(X), b \in \pi_k(X)$, the Whitehead product $[a, b]$ is in the kernel of the suspension homomorphism

$$\Sigma : \pi_{n+k-1}(X) \to \pi_{n+k}(\Sigma(X)).$$

9. Compute the homotopy group $\pi_3(S^2 \vee S^2)$.

10. (a) State the Poincaré duality theorem for manifolds with boundary.
(b) Let N be a compact connected oriented $4n$-dimensional manifold. The signature of N is defined to be the signature of the symmetric bilinear form[1]

$$H^{2n}(N, \mathbb{R}) \otimes H^{2n}(N, \mathbb{R}) \xrightarrow{\vee} H^{4n}(N, \mathbb{R}) = \mathbb{R}.$$

Prove that if N is the boundary of a compact orientable manifold M then the signature of N is zero. (Hint: show that there is a subspace $W \subset H^{2n}(N, \mathbb{R})$, with $2 \dim W = \dim H^{2n}(N, \mathbb{R})$, such that the restriction of the form to W is identically 0.)
(c) Show that $\mathbb{C}P^2 \times \mathbb{C}P^4$ is not homotopy equivalent to the boundary of a compact orientable manifold.

[1]The signature of a non-degenerate symmetric bilinear form on a real vector space is computed as follows: choose a basis in which the form is represented by a diagonal matrix. Then the signature is the number of positive diagonal entries minus the number of negative diagonal entries.