1. Prove that any continuous map $f: \mathbb{R}P^2 \to S^1$ is homotopy equivalent to a constant map.
2. Compute $\text{Tor}_*(\mathbb{Z}/8, \mathbb{Z}/6)$ and $\text{Ext}_*(\mathbb{Z}/8, \mathbb{Z}/6)$. 
3. (a) Calculate the degree of the map $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ given by $[z_0 : z_1] \mapsto [z_0^d : z_1^d]$.
(b) Define $g : \mathbb{C}P^2 \to \mathbb{C}P^2$ by $[z_0 : z_1 : z_2] \mapsto [z_0^d : z_1^d : z_2^d]$. Calculate the induced map $g^* : H^4(\mathbb{C}P^2, \mathbb{Z}) \to H^4(\mathbb{C}P^2, \mathbb{Z})$. 
4. (a) State the Lefschetz Fixed Point Theorem.
(b) Let $f: \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ be a map. Prove that $f$ always has a fixed point. Give an example that the above statement fails for a map $f: \mathbb{R}P^{2n+1} \to \mathbb{R}P^{2n+1}$. 
5. Let $M$ be a compact connected oriented surface of genus 2 i.e., a sphere with 2 handles.
(a) How many connected 2-fold covers $K \to M$ are there?
(b) For each such cover, find the genus of $K$. 
6. (a) State what it means for a manifold $M$ to be orientable.
(b) Let $G$ be a topological group, which is also a manifold. Prove that $G$ is orientable.
7. (a) Let $X$ be a topological space, $U, W \subset X$ open subsets. Give a definition of the relative cup product

$$H^p(X, U, \mathbb{Z}) \otimes H^q(X, W, \mathbb{Z}) \to H^{p+q}(X, U \cup W, \mathbb{Z}).$$

(b) Show that, for any space $Y$ and every $p, q > 0$, the cup product

$$H^p(\Sigma Y, \mathbb{Z}) \otimes H^q(\Sigma Y, \mathbb{Z}) \to H^{p+q}(\Sigma Y, \mathbb{Z})$$

is 0.
8. (a) For $a \in \pi_n(X), b \in \pi_k(X)$, define the Whitehead product $[a, b] \in \pi_{n+k-1}(X)$.
(b) Prove that, for any $X, a \in \pi_n(X), b \in \pi_k(X)$, the Whitehead product $[a, b]$ is in the kernel of the suspension homomorphism

$$\Sigma : \pi_{n+k-1}(X) \to \pi_{n+k}(\Sigma(X)).$$
9. Compute the homotopy group $\pi_3(S^2 \vee S^2)$. 
10. (a) State the Poincaré duality theorem for manifolds with boundary.
(b) Let $N$ be a compact connected oriented $4n$-dimensional manifold. The signature of $N$ is defined to be the signature of the symmetric bilinear form $^1$

$$H^{2n}(N, \mathbb{R}) \otimes H^{2n}(N, \mathbb{R}) \rightarrow H^{4n}(N, \mathbb{R}) = \mathbb{R}.$$ 

Prove that if $N$ is the boundary of a compact orientable manifold $M$ then the signature of $N$ is zero. (Hint: show that there is a subspace $W \subset H^{2n}(N, \mathbb{R})$, with $2\dim W = \dim H^{2n}(N, \mathbb{R})$, such that the restriction of the form to $W$ is identically 0.)

(c) Show that $\mathbb{C}P^2 \times \mathbb{C}P^4$ is not homotopy equivalent to the boundary of a compact orientable manifold.

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$^1$The signature of a non-degenerate symmetric bilinear form on a real vector space is computed as follows: choose a basis in which the form is represented by a diagonal matrix. Then the signature is the number of positive diagonal entries minus the number of negative diagonal entries.