1 Winter Differential Geometry 2016

Problem 1 *Prove that every bounded smooth vector field on* $\mathbb{R}^n$ *is complete.*

Problem 2 *Consider* $\left(\mathbb{R}^2, g\right)$ *to be the Riemannian manifold, with metric given by*

$$g = \left(e^{-x} + y^2 e^x\right) dx^2 + xye^{-x/2} dx dy + 10 \left(x^4 + y^4 + 5\right) dy^2.$$

1. *Argue that this is a Riemannian metric.*
2. *Is this a complete manifold? Prove or give a reason why it would not be.*

Problem 3 *Let*

$$\eta = f(x, y)dx + g(x, y)dy$$

be a closed one-form on $\mathbb{R}^2 \setminus \{0\}$ *and suppose that* f *and* g *are bounded. Prove that* η *is exact. Hint: Use some cohomology arguments that allow you to evaluate the nonexact parts of* η *along a standard path. You are free to draw a picture to describe a path if the picture is clear.*

Problem 4 *Let* (M, g) $(N, \tilde{g})$ *be Riemannian manifolds, and* $F : (M, g) \to (N, \tilde{g})$ *a smooth map. Consider the three following conditions* $Vol(M) > Vol(N)$

- F is locally conformal, i.e

$$\|DF(X)\|_{\tilde{g}} = \lambda \|X\|_g$$

  for all tangent vectors $X \in T_pM$.
- The conformal factor λ is a constant, with $\lambda > 1$.
- $Vol(M) > Vol(N)$.

Prove that F is not a diffeomorphism.

Problem 5

1. *True or False: Every simply connected 2 manifold is diffeomorphic to a flat Riemannian manifold. Prove or give counterexample.*
2. *True or False. Every simply connected 2 manifold with everywhere negative curvature is diffeomorphic to a flat Riemannian manifold. Prove or give counterexample.*

Problem 6 *Consider the one form*

$$\alpha = dx + zdy + zdz$$

on $\mathbb{R}^3$. *Show that there does not exist an embedding*

$$F : \mathbb{R}^2 \to \mathbb{R}^3$$

such that

$$F^*\alpha = 0.$$

Problem 7 *Suppose that a Riemmanian manifold has section curvatures of both* $+1$ *and* -1 *at a point* p*. Prove there exist a tangent plane at* p *that has zero sectional curvature.*

Problem 8 *Consider the manifold* $M^2 \subset \mathbb{R}^3$ *given by the solution*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

for some positive constants a, b, c*. Find the Gauss curvature at the point* $(0, b, 0)$.

Problem 9 *Consider the function*

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$

given by

$$f(a, b, c) = (a^2 + b^2 + c^2, 2a + b - c).$$

For what values of t *is*

$$X_t = f^{-1}(1, t)$$

a manifold? For each such t*, calculate the dimension of* X_t*.*