1 Fall Differential Geometry, 2015

Problem 1 *Recall that complex projective space* $\mathbb{C}P^2$ *is the space obtained as a quotient of* $\mathbb{C}^3 \setminus \{0\}$ *under the equivalence relation* $z \sim w$ *if there is* $\lambda \in \mathbb{C}$ *such that* $z = \lambda w$.

1. *Show that* $\mathbb{C}P^2$ *is a smooth manifold by exhibiting a smooth atlas for* $\mathbb{C}P^2$.
2. *Show that* $\mathbb{C}P^2$ *is compact and connected.*

Problem 2 *Give an expression for the Ricci curvature of the vector* ∂_1 *i.e*

$$Ric(\partial_1, \partial_1)$$

in terms of derivatives of Christoffel symbols at the origin of a normal coordinate system.

Problem 3 *Suppose that* (M^n, g) *is a open Riemannian manifold without boundary, and* f *is a smooth function with the property that*

$$\int_{\partial\Omega} \nabla f \cdot \vec{n} d\tilde{V} > 0$$

for every set Ω *that is diffeomorphic to a closed ball, where* $\vec{n}$ *is the exterior normal and* $d\tilde{V}$ *is the volume form on the boundary* $\partial\Omega$. *Prove that*

$$\Delta_g f(x) \geq 0 \text{ for all } x \in M.$$

Problem 4 *Suppose* (M, g) *and* (N, h) *are two Riemannian manifolds, which are diffeomorphic. Which of the following must be shared by both* M *and* N? *Give a quick (2 to 3 sentence/line) proof, or provide a counterexample.*

1. *Compactness*
2. *Completeness*
3. *Sectional curvature lower bounds*
4. *Dimension of the manifold.*
5. *Triviality of the tangent bundle.*

Problem 5 *Prove that any compact manifold may be smoothly embedded in* $\mathbb{R}^N$ *for some large* N. *Your proof should be essentially self-contained, not citing significant theorems.*

Problem 6 *Let* $f : M \to \mathbb{R}$ *be a smooth function on a smooth manifold, and let* $p \in M$ *be a critical point of* f. *A critical point is called a nondegenerate critical point if the matrix of second derivatives of* f *at* p,*with respect to a local coordinate system, is nonsingular. Prove that this notion of nondegeneracy is independent of the choice of coordinate system*

Problem 7 *Consider the space* $Y = L^2(\mathbb{R})\backslash\{0\}$.

1. Give a 'cocktail party' argument that Y is a "Banach Manifold"
2. Consider the space

$$X = \left\{ f \in L^2(\mathbb{R}) : \|f\|_{L^2}^2 = 1 \right\}.$$

   Give an argument that X is also "Banach manifold", by giving a chart from a neighborhood of any point in X to a linear Banach space.

Problem 8 *Let* $M^2 \subset \mathbb{R}^3$ *be the graph of a function* f. *Suppose that* $df = 0$ *at* 0.

1. Write down a local tangent frame for the manifold M near 0 (the expression should be in terms of vectors in $\mathbb{R}^3$)
2. Prove that the mean curvature satisfies

$$H(0) = \frac{1}{2}\Delta f(0).$$

Problem 9 *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *satisfy*

$$|df|^2 \leq \frac{x^2 + y^2}{1 + x^2 + y^2}.$$

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ *be the map*

$$F(x, y) = (x, y, f(x, y)).$$

Now consider the Minkowski metric

$$h = dx^2 + dy^2 - dz^2$$

on $\mathbb{R}^3$. *Is the image of* F *necessarily a complete manifold in* $(\mathbb{R}^3, h)$?