Problem 1 Recall that complex projective space $\mathbb{C}P^2$ is the space obtained as a quotient of $\mathbb{C}^3 \setminus \{0\}$ under the equivalence relation $z \sim w$ if there is $\lambda \in \mathbb{C}$ such that $z = \lambda w$.

1. Show that $\mathbb{C}P^2$ is a smooth manifold by exhibiting a smooth atlas for $\mathbb{C}P^2$.
2. Show that $\mathbb{C}P^2$ is compact and connected.

Problem 2 Give an expression for the Ricci curvature of the vector $\partial_1$ i.e.

$$\text{Ric}(\partial_1, \partial_1)$$

in terms of derivatives of Christoffel symbols at the origin of a normal coordinate system.

Problem 3 Suppose that $(M^n, g)$ is a open Riemannian manifold without boundary, and $f$ is a smooth function with the property that

$$\int_{\partial \Omega} \nabla f \cdot \vec{n} d\tilde{V} > 0$$

for every set $\Omega$ that is diffeomorphic to a closed ball, where $\vec{n}$ is the exterior normal and $d\tilde{V}$ is the volume form on the boundary $\partial \Omega$. Prove that

$$\Delta_g f(x) \geq 0$$

for all $x \in M$.

Problem 4 Suppose $(M, g)$ and $(N, h)$ are two Riemannian manifolds, which are diffeomorphic. Which of the following must be shared by both $M$ and $N$? Give a quick (2 to 3 sentence/line) proof, or provide a counterexample.

1. Compactness
2. Completeness
3. Sectional curvature lower bounds
4. Dimension of the manifold.
5. Triviality of the tangent bundle.

Problem 5 Prove that any compact manifold may be smoothly embedded in $\mathbb{R}^N$ for some large $N$. Your proof should be essentially self-contained, not citing significant theorems.

Problem 6 Let $f : M \to \mathbb{R}$ be a smooth function on a smooth manifold, and let $p \in M$ be a critical point of $f$. A critical point is called a nondegenerate critical point if the matrix of second derivatives of $f$ at $p$, with respect to a local coordinate system, is nonsingular. Prove that this notion of nondegeneracy is independent of the choice of coordinate system.
**Problem 7** Consider the space \( Y = L^2(\mathbb{R}) \setminus \{0\} \).

1. Give a 'cocktail party' argument that \( Y \) is a "Banach Manifold"

2. Consider the space

\[
X = \left\{ f \in L^2(\mathbb{R}) : \|f\|_{L^2}^2 = 1 \right\}.
\]

Give an argument that \( X \) is also "Banach manifold", by giving a chart from a neighborhood of any point in \( X \) to a linear Banach space.

**Problem 8** Let \( M^2 \subset \mathbb{R}^3 \) be the graph of a function \( f \). Suppose that \( df = 0 \) at 0.

1. Write down a local tangent frame for the manifold \( M \) near 0 (the expression should be in terms of vectors in \( \mathbb{R}^3 \))

2. Prove that the mean curvature satisfies

\[
H(0) = \frac{1}{2} \Delta f(0).
\]

**Problem 9** Let \( f : \mathbb{R}^2 \to \mathbb{R} \) satisfy

\[
|df|^2 \leq \frac{x^2 + y^2}{1 + x^2 + y^2}.
\]

Let \( F : \mathbb{R}^2 \to \mathbb{R}^3 \) be the map

\[
F(x, y) = (x, y, f(x, y)).
\]

Now consider the Minkowski metric

\[
h = dx^2 + dy^2 - dz^2
\]

on \( \mathbb{R}^3 \). Is the image of \( F \) necessarily a complete manifold in \( (\mathbb{R}^3, h) \)?