Fall 2015 Probability Qualifying Exam

September 21, 2015

Instructions:

1. Read all questions carefully. If you are confused ask me!
2. You should have 3 pages including this page. Make sure you have the right number of pages.
3. Unless otherwise noted all conventions and notation follow that of Çinlar. If you are confused about notation, ask!
4. If necessary you may use the back of pages.

Name:______________________________

UO ID:______________________________

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	10	10	10	10	10	10	10	80
Score:									

[10 pts] 1. (a) Suppose Z is a real valued random variable. Show that Z is integrable if and only if

$$\lim_{b\to\infty} E[|Z|\mathbf{1}_{|Z|>b}] = 0.$$

(b) Suppose $\mathcal{K}$ is a collection of real-valued random variables, such that every $X \in \mathcal{K}$ is dominated by an integrable random variable Z ($|X| \leq Z$ for all $X \in \mathcal{K}$). Show that $\mathcal{K}$ is uniformly integrable.

[10 pts] 2. State and prove Kolmogorov's 0-1 Law. Make sure to include all relevant definitions (*e.g.* tail σ-algebra etc.).

[10 pts] 3. Let $B_1, B_2, \ldots$ be Bernoulli random variables (not necessarily identically distributed).

(a) Show that $\sum_n E[B_n] < \infty$ implies $\sum_n B_n < \infty$ almost surely.

(b) Show that if the (B_n) are pairwise independent then $\sum_n E[B_n] = +\infty$ implies $\sum_n B_n = +\infty$ almost surely.

[10 pts] 4. (a) Prove that convergence in L^2 implies convergence in probability.

(b) Suppose (X_n) is a sequence of identically distributed, pairwise independent random variables with finite mean a and finite variance b. Show that $\lim_n \frac{1}{n}(X_1 + \cdots + X_n)$ converges to $E[X_n] = a$ in probability.

[10 pts] 5. Let $X : [0,1] \to [0,1]$ be given by

$$X(t) = \begin{cases} 2t & \text{if } t < 1/2; \\ 1/2 & \text{if } t \geq 1/2. \end{cases}$$

Assume that $[0,1]$ is equipped with the Borel σ-algebra and $\mathbb{P}$ is Lebesgue measure.

(a) Describe σX. (An explicit description of a p-system which generates σX is acceptable).

(b) Let $Y(t) = t^2$ defined on $[0,1]$. Compute $E_{\sigma X}[Y]$ and prove your answer is a version of (i.e. almost surely) the conditional expectation.

[10 pts] 6. Let K be a probability transition kernel from measurable spaces $(D, \mathcal{D})$ into $(E, \mathcal{E})$, and suppose μ is a measure on $(D, \mathcal{D})$. Let X and Y be random variables in $(D, \mathcal{D})$ and $(E, \mathcal{E})$ respectively, with joint density given by

$$\pi(dx, dy) = \mu(dx)K(x, dy).$$

Show that a version of the conditional probability of Y given $\mathcal{F} = \sigma X$ is given by $K(X, B)$ where $B \in \mathcal{E}$.

[10 pts] 7. Let $T_1 < T_2 < \ldots,$ be random times in $\mathbb{T} = \mathbb{R}_+$ with $\lim_n T_n = +\infty$, and let $N = (N_t)_{t\in\mathbb{T}}$ be given by

$$N_t = \sum_{n=1}^{\infty} \mathbf{1}_{[0,t]} \circ T_n.$$

Let $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$ be the filtration generated by N.

Question 7 continues...

(a) Let $T = \inf\{t : N_t \geq k\}$. Is T a stopping time? Explain.

(b) Fix b. Let $L = \inf\{t : N_t = N_b\}$. Is L a stopping time? Explain.

(c) Define the process A by $A_t(\omega) = t - T_k(\omega)$ if $T_k(\omega) \leq t < T_{k+1}(\omega)$. Show that A is adapted to $\mathcal{F}$.

[10 pts] 8. Let $W = (W_t)_{t \in \mathbb{R}_+}$ be a continuous process with respect to the filtration $\mathcal{F}$. Show that if

$$M_t = \exp\left\{rW_t - \frac{1}{2}r^2 t\right\}; \qquad t \in \mathbb{R}_+$$

is an $\mathcal{F}$-martingale, then W is a Wiener process.

End of Exam

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