Qualifying Exam in Algebra
Winter 2016

Part I. Definitions and theorems.
1. (4 points). Define adjoint functors and describe the relation between adjoint functors and categorical limits/colimits.
2. (4 points). Define the character of a finite-dimensional representation of a finite group and state two orthogonality relations for irreducible complex characters.
3. (4 points). Formulate the Wedderburn-Artin Theorem.

Part II. True or false? Give a brief justification.
1. (8 points). If a finite group admits an irreducible complex representation of dimension 3 then it has at least 12 elements.
2. (8 points). If $F/K/k$ are finite field extensions with F/K and K/k normal then F/k is also normal.
3. (8 points). Let I and J be radical ideals of $\mathbb{C}[x, y]$. Then $I + J$ is also radical.
4. (8 points). If R is a left artinian ring with no zero divisors then R is a division ring.

Part III. Solve the following problems.
1. (12 points). Let P be a non-maximal prime ideal of $\mathbb{Z}[x]$. Prove that P is a principal ideal.
2. (12 points). Let G be a group of order $2^3 \cdot 7^2 \cdot 11$. Show that G has a subgroup of order 77.
3. (12 points). Let R be an integral domain with the field of fractions F. Prove that F is a flat and injective R-module.
4. (12 points). Let K be a field and assume that -1 is not a square in K. Show that an element $g \in \mathrm{GL}_2(K)$ has order 4 if and only if $\det(g) = 1$ and $\mathrm{Tr}(g) = 0$.