Qualifying Exam in Algebra

Fall 2015

Part I. Definitions and theorems.

1. (4 points). State the fundamental theorem of Galois theory.
2. (4 points). Give a definition of the Jacobson radical of a ring. What can you say about the Jacobson radical of a left artinian ring?
3. (4 points). State the Jordan-Hölder Theorem.

Part II. True or false? Give a brief justification.

1. (8 points). Let A be an $n \times n$ matrix with rational coefficients such that $A^7 - 5A^3 + A = 0$. Then $\mathrm{Tr}(A^k)$ is an integer for every $k \geq 0$.
2. (8 points). If R is a PID and A is a ring which is Morita equivalent to R then A is isomorphic to the matrix ring $M_n(R)$ for some n.
3. (8 points). If a ring has a finite number of irreducible modules up to isomorphism then it is left artinian.
4. (8 points). There are no simple groups of order 200.

Part III. Solve the following problems.

1. (12 points). Let M be a maximal ideal in $F[x_1, \ldots .x_n]$, where F is a field. Prove that there exists a finite field extension $F \subset E$ and an n-tuple $(a_1, \ldots, a_n) \in E^n$ such that

$$M = \{f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n] \mid f(a_1, \ldots, a_n) = 0\}.$$

2. (12 points). Let $\mathcal{C}$ be the category of quadruples (V_1, V_2, f_1, f_2), where V_1 and V_2 are $\mathbb{C}$-vector spaces and $f_1, f_2 : V_1 \to V_2$ are linear maps, with morphisms defined in an obvious way (so $\mathcal{C}$ is the category of complex representations of the quiver with vertices 1 and 2 and two arrows from 1 to 2). Consider the functor F from the category Vect of $\mathbb{C}$-vector spaces to $\mathcal{C}$ sending $V \in \mathrm{Vect}$ to $F(V) = (V, V, \mathrm{id}_V, \mathrm{id}_V) \in \mathcal{C}$. Prove that F admits the left adjoint and the right adjoint functors and describe them explicitly.
3. (12 points). Let R be a ring with 1 and let

$$0 = I_0 \subset I_1 \subset \ldots \subset I_n = R$$

be a chain of right ideals of R such that the n quotients $V_i = I_i/I_{i-1}$ are pairwise nonisomorphic simple right R-modules. Prove that the ring R is semisimple.

4. (12 points). Let F be the finite field with q elements, and let

$$G = \{f : F \to F \ : \ f(x) = ax + b, a \in F^*, \ b \in F\}$$

be the group of affine transformations of F. Let V be the permutation representation of G (over $\mathbb{C}$) corresponding to the natural action of G on F. Prove that V contains the trivial representation $\mathbf{1}$ and that the quotient $V/\mathbf{1}$ is irreducible. Find all irreducible representations of G.