Qualifying Exam in Algebra  
Fall 2015

**Part I.** Definitions and theorems.
1. (4 points). State the fundamental theorem of Galois theory.
2. (4 points). Give a definition of the Jacobson radical of a ring. What can you say about the Jacobson radical of a left artinian ring?
3. (4 points). State the Jordan-Hölder Theorem.

**Part II.** True or false? Give a brief justification.
1. (8 points). Let $A$ be an $n \times n$ matrix with rational coefficients such that $A^7 - 5A^3 + A = 0$. Then $\text{Tr}(A^k)$ is an integer for every $k \geq 0$.
2. (8 points). If $R$ is a PID and $A$ is a ring which is Morita equivalent to $R$ then $A$ is isomorphic to the matrix ring $M_n(R)$ for some $n$.
3. (8 points). If a ring has a finite number of irreducible modules up to isomorphism then it is left artinian.
4. (8 points). There are no simple groups of order 200.

**Part III.** Solve the following problems.
1. (12 points). Let $M$ be a maximal ideal in $F[x_1, \ldots, x_n]$, where $F$ is a field. Prove that there exists a finite field extension $F \subset E$ and an $n$-tuple $(a_1, \ldots, a_n) \in E^n$ such that
   $$M = \{ f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n] \mid f(a_1, \ldots, a_n) = 0 \}.$$  
2. (12 points). Let $C$ be the category of quadruples $(V_1, V_2, f_1, f_2)$, where $V_1$ and $V_2$ are $C$-vector spaces and $f_1, f_2 : V_1 \to V_2$ are linear maps, with morphisms defined in an obvious way (so $C$ is the category of complex representations of the quiver with vertices 1 and 2 and two arrows from 1 to 2). Consider the functor $F$ from the category Vect of $C$-vector spaces to $C$ sending $V \in \text{Vect}$ to $F(V) = (V, V, \text{id}_V, \text{id}_V) \in C$. Prove that $F$ admits the left adjoint and the right adjoint functors and describe them explicitly.
3. (12 points). Let $R$ be a ring with 1 and let
   $$0 = I_0 \subset I_1 \subset \ldots \subset I_n = R$$
be a chain of right ideals of $R$ such that the $n$ quotients $V_i = I_i/I_{i-1}$ are pairwise nonisomorphic simple right $R$-modules. Prove that the ring $R$ is semisimple.
4. (12 points). Let $F$ be the finite field with $q$ elements, and let
   $$G = \{ f : F \to F : f(x) = ax + b, a \in F^*, b \in F \}$$
be the group of affine transformations of $F$. Let $V$ be the permutation representation of $G$ (over $\mathbb{C}$) corresponding to the natural action of $G$ on $F$. Prove that $V$ contains the trivial representation $1$ and that the quotient $V/1$ is irreducible. Find all irreducible representations of $G$. 