ANALYSIS QUALIFYING EXAM, FALL 2015

1. Determine, with justification, the limit

$$\lim_{n\to\infty} \int_0^n \frac{\cos(x/n)}{x^2 + \cos(x/n)}\, dx.$$

2. Prove that for any Lebesgue measurable function $f\colon \mathbb{R} \to \mathbb{C}$ and any $\varepsilon > 0$, there exists a continuous function $g\colon \mathbb{R} \to \mathbb{C}$ such that the set

$$\{x \in \mathbb{R}\colon f(x) \neq g(x)\}$$

has Lebesgue measure less than ε.

3. Suppose that $f, g \in L^1(\mathbb{R})$. For $n \in \mathbb{N}$, define $g_n \in L^1(\mathbb{R})$ by $g_n(x) = g(x - n)$ for $x \in \mathbb{R}$. Show that

$$\lim_{n\to\infty} \|f + g_n\|_1 = \|f\|_1 + \|g\|_1.$$

4. Suppose that K is a compact metric space and $Y \subset C(K)$ is a closed subspace. Let $f \in C(K)$. Suppose that for every complex Borel measure μ on K such that $\int_K g\, d\mu = 0$ for all $g \in Y$, we have $\int_K f\, d\mu = 0$. Show that $f \in Y$.

5. Let X be a normed space and let Y be a Banach space. Let $L(X, Y)$ be the normed space of all linear maps from X to Y with the usual operator norm. Prove that $L(X, Y)$ is complete.

6. Let $(e_n)_{n\in\mathbb{Z}}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Given a sequence $(\alpha_n)_{n\in\mathbb{Z}}$ in $\mathbb{R}$, define elements $x_n \in \mathcal{H}$ by

$$x_{2n} = \cos(\alpha_n)e_{2n} - \sin(\alpha_n)e_{2n+1} \qquad \text{and} \qquad x_{2n+1} = \sin(\alpha_n)e_{2n} + \cos(\alpha_n)e_{2n+1}$$

for $n \in \mathbb{Z}$. Show that $(x_n)_{n\in\mathbb{Z}}$ is also an orthonormal basis of $\mathcal{H}$.

7. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^2(\mathbb{R})$. Suppose that for all $n \in \mathbb{N}$, the Fourier transform $\widehat{f_n}$ vanishes outside $[n,\, n+1]$ and that

$$\|f_n\|_2 = n^{-\frac{51}{100}}.$$

Show that the series $\sum_{n=1}^{\infty} f_n$ converges in norm in $L^2(\mathbb{R})$.

8. Define a subset $X \subset \mathbb{C}$ by $X = \{i/n\colon n \in \mathbb{N}\}$. Let $\overline{X}$ denote its closure. Suppose that f is a bounded holomorphic function on $\mathbb{C} \setminus \overline{X}$. Prove that f is constant.

9. Define $f\colon \mathbb{C} \to \mathbb{C}$ by $f(z) = z^{2015} - 2z^{15} + 15z^2 + 7i$ for $z \in \mathbb{C}$. How many zeros (counting multiplicity) does f have in the open disk centered at 0 of radius 2?