Please note: typo in the year. Should read 2015

ALGEBRA QUALIFYING EXAMINATION, WINTER 2014

Your Name:

Part I. Carefully state each of the following (6 points each):

1. Describe what is Yoneda's Embedding.

2. Maschke's Theorem.

3. Hilbert's Basis Theorem.

Part II. True or false? If true provide a brief explanation, if false provide a counterexample (8 points each):

1. Let R be a ring and I, J be two left ideals in R. If the modules R/I and R/J are isomorphic then $I = J$.

2. Let G be a finite group such that every irreducible $\mathbb{C}G$-module is one-dimensional. Then G is abelian.

3. A domain R is a field if and only if every R-module is projective.

4. Let R be a noetherian local ring with maximal ideal M and let $x_1, \dots, x_n \in M$. If $\{x_1 + M^2, \dots, x_n + M^2\}$ is a basis of the R/M-vector space M/M^2, then $M = Rx_1 + \cdots + Rx_n$.

5. $\{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 = 1\}$ is homeomorphic to $\mathbb{C}$ in Zariski topology.

Part III. Prove the following statements (14 points each):

1. Let G be a finite group, $H \leq G$, $V \in \mathbb{C}H$-mod and $W \in \mathbb{C}G$-mod. Then there is an isomorphism of $\mathbb{C}G$-modules
$$(\operatorname{ind}_H^G V) \otimes W \cong \operatorname{ind}_H^G (V \otimes \operatorname{res}_H^G W).$$

2. Let K/F be a field extension and $A, B \in M_n(F)$. If A and B are similar as matrices over K then they are similar as matrices over F.

3. Let R, R' be rings and $\mathcal{F} : R\text{-}\mathtt{Mod} \to R'\text{-}\mathtt{Mod}$ be a functor left adjoint to a functor $\mathcal{G} : R'\text{-}\mathtt{Mod} \to R\text{-}\mathtt{Mod}$.

(a) Prove that $\mathcal{G}$ is left exact.

(b) If $\mathcal{G}$ is exact and P is a projective R-module, then $\mathcal{F}P$ is a projective R'-module.