Q1. Let \( f \) be a nonnegative measurable function on \([0, 1]\) with Lebesgue measure \( m \).
   (1a) Prove that \( \int \int_{[0, 1]} f(x) \, dm(x) \leq \sum_{n=0}^{\infty} m(\{x, f(x) \geq n\}) \).
   (1b) Assume \( m(\{x, f(x) \geq t\}) \leq \frac{1}{1+t^2} \) for each \( t > 0 \). Prove that \( f \in L^p \) for \( p \in [1, 2) \).

Q2. Let \( A : X \to X \) be a linear operator on complex normed space \( X \). Assume \( \lambda \) is an eigenvalue of \( A^n \circ A \circ \cdots \circ A \) for some integer \( n \geq 2 \). Prove that one of the complex \( n \)-th roots of \( \lambda \) is an eigenvalue of \( A \).

Q3. Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of measurable functions on \([0, \pi]\) satisfying
   \[ \int_0^{\pi} |f_n(x)|^2 \, dm(x) \leq 2015, \]
   where \( m \) is the Lebesgue measure. Suppose \( f_n \to 0 \) a.e. on \([0, \pi]\). Prove that
   \[ \int_0^{\pi} |f_n(x)| \, dm(x) \to 0. \]
   (Hint: Use Egorov’s theorem and the Cauchy-Schwarz inequality.)

Q4. Let \((X, \mu)\) be a measure space and let \( f \in L^k(\mu) \) for some real number \( k \geq 1 \). Compute
   \[ \lim_{n \to \infty} \int_X n^k \ln \left( 1 + \left( \frac{|f|}{n} \right)^k \right) \, d\mu. \]
   (Hint: First show that there is a constant \( C > 0 \) such that \( \ln(1 + y) \leq Cy \) for \( y \in [0, \infty) \).)

Q5. Note that \( \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \, dx = \pi \). Suppose \( f(x) \) is a bounded continuous function on \( \mathbb{R} \). Define for each \( \sigma \in (0, \infty) \)
   \[ f_{\sigma}(x) := \frac{1}{\pi} \int_{-\infty}^{\infty} f \left( x + \frac{2\sigma t}{\sigma} \right) \frac{\sin^2 \sigma t}{\sigma^2} \, dt. \]
   Prove that on any finite interval \( x \in [a, b] \), the functions \( f_{\sigma}(x) \) converge uniformly to \( f(x) \) as \( \sigma \to +\infty \).
Q6. Let $A$ be a bounded linear operator on real Hilbert space $H$. Recall that the adjoint operator $A^*$ is defined by $(Ax, y) = (x, A^*y)$ for $x, y \in H$.

(6a) Show that norm $\|A^*\| = \|A\|$.  
(6b) Show that norm $\|A^*A\| = \|AA^*\| = \|A\|^2$.

Q7. Find the Laurent expansion of the function

$$f(z) = \frac{1}{(z + 1)(z + 2)}$$

which holds in $2 < |z - 1| < 3$.

Q8. Let $f(z)$ be a polynomial of degree 2015. Prove that the sum of the residues of $\frac{1}{f(z)}$ at all the zeros of $f(z)$ must be zero.

Q9. Compute

$$\int_0^\infty \frac{x^\alpha}{x^2 + x + 1} \, dx$$

where $\alpha$ is a real number satisfying $0 < \alpha < 1$. 
