ANALYSIS QUALIFYING EXAM, Winter 2015

NAME:

STUDENT NUMBER: SIGNATURE:

Q1. Let f be a nonnegative measurable function on $[0, 1]$ with Lebesgue measure m.
(1a) Prove that $\int_{[0,1]} f(x)dm(x) \leq \sum_{n=0}^{\infty} m(\{x,\ f(x) \geq n\})$.
(1b) Assume $m(\{x,\ f(x) \geq t\}) \leq \frac{1}{1+t^2}$ for each $t > 0$. Prove that $f \in L^p$ for $p \in [1, 2)$.

Q2. Let $A : X \to X$ be a linear operator on complex normed space X. Assume λ is an eigenvalue of $A^n \doteqdot A \circ A \circ \cdots \circ A$ for some integer $n \geq 2$. Prove that one of the complex n-th roots of λ is an eigenvalue of A.

Q3. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on $[0, \pi]$ satisfying

$$\int_0^{\pi} |f_n(x)|^2 dm(x) \leq 2015,$$

where m is the Lebesgue measure. Suppose $f_n \to 0$ a.e. on $[0, \pi]$. Prove that

$$\int_0^{\pi} |f_n(x)| dm(x) \to 0.$$

(Hint: Use Egorov's theorem and the Cauchy-Schwarz inequality.)

Q4. Let (X, μ) be a measure space and let $f \in L^k(\mu)$ for some real number $k \geq 1$. Compute

$$\lim_{n\to\infty} \int_X n^k \ln\left(1 + \left(\frac{|f|}{n}\right)^k\right) d\mu.$$

(Hint: First show that there is a constant $C > 0$ such that $\ln(1 + y) \leq Cy$ for $y \in [0, \infty)$.)

Q5. Note that $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi$. Suppose $f(x)$ is a bounded continuous function on $\mathbb{R}$. Define for each $\sigma \in (0, \infty)$

$$f_\sigma(x) \doteqdot \frac{1}{\pi} \int_{-\infty}^{\infty} f\left(x + \frac{2t}{\sigma}\right) \frac{\sin^2 t}{t^2} dt.$$

Prove that on any finite interval $x \in [\alpha, \beta]$, the functions $f_\sigma(x)$ converge uniformly to $f(x)$ as $\sigma \to +\infty$.

Q6. Let A be a bounded linear operator on real Hilbert space H. Recall that the adjoint operator A^* is defined by $(Ax, y) = (x, A^*y)$ for $x, y \in H$.

(6a) Show that norm $\|A^*\| = \|A\|$.

(6b) Show that norm $\|A^*A\| = \|AA^*\| = \|A\|^2$.

Q7. Find the Laurent expansion of the function

$$f(z) = \frac{1}{(z+1)(z+2)}$$

which holds in $2 < |z-1| < 3$.

Q8. Let $f(z)$ be a polynomial of degree 2015. Prove that the sum of the residues of $\frac{1}{f(z)}$ at all the zeros of $f(z)$ must be zero.

Q9. Compute

$$\int_0^\infty \frac{x^\alpha}{x^2+x+1} dx$$

where α is a real number satisfying $0 < \alpha < 1$.