Analysis Qualifying Exam, Fall 2014

NAME:

STUDENT NUMBER: **SIGNATURE**:

Q1. Let μ^* be the Lebesgue outer measure on $\mathbb{R}$. Let A and B be two "not necessarily measurable" subsets of $\mathbb{R}$ satisfying

$$\inf\{|x-y|,\, x \in A, y \in B\} > 0.$$

Prove that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

Q2. Let A be a bounded Lebesgue measurable set in $\mathbb{R}$. Let $\{c_n\}$ be a sequence in $\mathbb{R}$. Prove that

$$\lim_{n\to\infty} \int_A \sin^2(nx + c_n)dm(x) = \frac{1}{2}m(A),$$

where m is the Lebesgue measure.

Q3. Let $f_n(x)$ for $n \in \mathbb{N}$ and $f_\infty(x)$ be integrable functions on the measure space (X, μ). Suppose $f_n(x) \to f_\infty(x)$ μ-a.e. and suppose

$$\int_X |f_n(x)|d\mu(x) \to \int_X |f_\infty(x)|d\mu(x).$$

Use Fatou's lemma to prove that for any measurable set $A \subset X$

$$\int_A |f_n(x)|d\mu(x) \to \int_A |f_\infty(x)|d\mu(x).$$

Q4. Let $\{a_n\}_{n=2}^\infty$ be a sequence of real numbers satisfying $|a_n| \le (\ln n)^\alpha$ for some $\alpha > 0$. Consider series

$$\sum_{n=2}^\infty a_n n^{-x}, \quad \text{for } x \in [2, \infty).$$

4a) Prove that this series converges in $L^1[2, \infty)$.

4b) Prove that

$$\sum_{n=2}^\infty \int_2^\infty a_n n^{-x} dx = \int_2^\infty \sum_{n=2}^\infty a_n n^{-x} dx.$$

Q5. Let $(X_1, \|\cdot\|_1), (X_2, \|\cdot\|_2), \cdots$ be a sequence of real Banach spaces. Let real number $p \geq 1$. We define a set Y by

$$Y \doteqdot \left\{ \vec{x} \doteqdot (x_1, x_2, \cdots) \in X_1 \times X_2 \times \cdots,\ \sum_{n=1}^{\infty} \|x_n\|_n^p < \infty \right\}.$$

Y is given a vector space structure with the natural scalar product and addition. On Y we define

$$\|\vec{x}\|_Y \doteqdot \left(\sum_{n=1}^{\infty} \|x_n\|_n^p \right)^{1/p}.$$

Prove that $(Y, \|\cdot\|_Y)$ is a real Banach space. (You do **not** need to show that Y is a vector space. Recall that verifying completeness is a major step)

Q6. Let $f(z)$ be a degree n polynomial and for any real number $R > 0$, let $M(R) \doteqdot \max_{|z|=R} |f(z)|$. Show that if $R_2 > R_1 > 0$, then

$$\frac{M(R_2)}{R_2^n} \leq \frac{M(R_1)}{R_1^n},$$

with equality being possible only if $f(z) = Cz^n$, for some constant $C \in \mathbb{C}$.

Q7. Compute the path integral

$$\int_{|z|=1} \frac{e^{1/z}}{1-5z} dz.$$

(Hint: since $z = 0$ is a singularity of $e^{1/z}$, you may consider a change of variable first.)

Q8. Compute

$$\int_0^{\infty} \frac{\ln x}{x^2 + b^2} dx$$

where b is a positive real number.

Q9. Let $T : X \to Y$ be bounded linear operator between normed spaces.

9a) Prove that for any $x, w \in X$ we have

$$\max\{\|T(x+w)\|, \|T(x-w)\|\} \geq \|T(x)\|. \tag{1}$$

9b) Prove that for any $x \in X$ and $r > 0$

$$\sup_{\tilde{x} \in B(x,r)} \|T(\tilde{x})\| \geq r\|T\|. \tag{2}$$

where $B(x, r)$ is the open ball in X centered at x with radius r.

9c) Let $T_n : X \to Y$, $n \in \mathbb{N}$, be a family of bounded linear operators. Suppose $\|T_n\| \geq 4^n$. By 9b) there are $x_0, x_1, x_2, \dots \in X$ with $x_0 = 0$ such that $\|x_n - x_{n-1}\| \leq 3^{-n}$ and $\|T_n x_n\| \geq \frac{2}{3} 3^{-n} \|T_n\|$. Use this to prove that when X is a Banach space, then there is a $\hat{x} \in X$ such that $\|T_n \hat{x}\| \to \infty$.
(Note that this gives another proof of Banach-Steinhaus theorem.)