TOPOLOGY QUALIFYING EXAM, UNIVERSITY OF OREGON, FALL 2015

Each problem is worth 10 points. When proving statements, you may use results from Hatcher's and May's textbooks or from Botwinnik's lecture notes. Do not forget to sign your paper. Good luck!

1. Consider the map $p : \mathbb{C}^{n+1} - 0 \to \mathbb{C}P^n$, which takes a vector $x \in \mathbb{C}^{n+1} - 0$ to the line $\mathbb{C}x$. Prove that, for $n > 0$, there is no continuous map $s : \mathbb{C}P^n \to \mathbb{C}^{n+1} - 0$ such that $p \circ s = Id$.

2. Compute the cohomology ring of $H^*(X, \mathbb{Z}/2)$, for $X = \mathbb{R}P^n - \{x_1, x_2\}$, where $x_1 \neq x_2$ are any two points in $\mathbb{R}P^n$.

3. Calculate the degree of the map $x \mapsto -x$ on the space S^n.

4. (a) State the Lefschetz Fixed Point Theorem.
(b) Prove that every map $f : \mathbb{C}P^{2n} \times \mathbb{R}P^2 \times \mathbb{R}P^4 \to \mathbb{C}P^{2n} \times \mathbb{R}P^2 \times \mathbb{R}P^4$ has a fixed point.

5. (a) Define the Hopf invariant $h(\lambda)$ for $\lambda \in \pi_{4q-1}(S^{2q})$.
(b) Show that there is an element $\lambda \in \pi_7(S^4)$ with $h(\lambda) = 1$. (Hint: look at the quaternionic projective space $\mathbb{H}P^2$).

6. Let $X = (S^3 \times S^1)/\sim$ where $(x, y) \sim (-x, -y)$ for $x \in S^3, y \in S^1$.
(a) Calculate $\pi_i(X)$ for $1 \leq i \leq 3$.
(b) Show that $p : X \to \mathbb{R}P^3$ given by p([(x,y)]) = [x] is a fiber bundle with fiber S^1.
(c) Is X homeomorphic to $S^1 \times \mathbb{R}P^3$? Explain.

7. (a) State the the Poincaré duality theorem for compact orientable manifolds.
(b) Let M be a compact orientable manifold of dimension 3. Prove that $H_2(M, \mathbb{Z})$ has no torsion.

8. (a) Let $K \subset S^n$ be a locally contractible closed subset. State the Alexander duality theorem relating $\hat{H}^m(K, \mathbb{Z})$ and $\hat{H}_k(S^n - K, \mathbb{Z})$.
(b) Prove that $\mathbb{C}P^2 \times \mathbb{R}P^2$ can not be embedded in $\mathbb{R}^7$. (Hint: assuming the contrary, $\mathbb{C}P^2 \times \mathbb{R}P^2 \subset \mathbb{R}^7 = S^7 - \infty$ compute $\hat{H}_0(S^7 - \mathbb{C}P^2 \times \mathbb{R}P^2, \mathbb{Z})$.)

9. (a) State the Hurewicz theorem and its relative version.
(b) Let X be a $(n-1)$-connected space, $n \geq 2$. Prove that the homomorphism $\pi_{n+1}(X) \to H_{n+1}(X, \mathbb{Z})$ is surjective.

10. (a) Let $p : E \to B$ be a fiber bundle with fiber F. Define the connecting homomorphism $d : \pi_k(B) \to \pi_{k-1}(F)$.
(b) Let $SU(n)$ be the special unitary group (*i.e.*, the group of complex, unitary $n \times n$ matrices with determinant equal to 1). Compute $\pi_i(SU(n))$ and $H_i(SU(n))$ for $i \leq 3$.