ANALYSIS QUALIFYING EXAM, WINTER 2014

1. Let $(X, \mathfrak{M}, \mu)$ be a positive measure space and let $f \in L^1(\mu)$. For $n \in \mathbb{Z}_{>0}$ define

$$E_n = \{x \in X : |f(x)| \geq n\}.$$

(1) Prove that $\lim_{n\to\infty} \mu(E_n) = 0$.
(2) Prove that $\lim_{n\to\infty} n\mu(E_n) = 0$.

2. Let $(X, \mathfrak{M}, \mu)$ be a finite positive measure space. Let $p \in (0, \infty)$, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^p(\mu)$. Assume that:
(1) $f_n \to f$ almost everywhere on X as $n \to \infty$.
(2) $\|f_n\|_p \to 0$ as $n \to \infty$.
Prove that $\|f_n - f\|_p \to 0$ as $n \to \infty$.

3. Let m be Lebesgue measure on $\mathbb{R}$. Prove that there is no measurable subset $E \subset \mathbb{R}$ such that

$$1/3 < \mu(E \cap (a, b)) < 2/3$$

for all $a, b \in \mathbb{R}$ with $a < b$.

4. Let $(X, \mathfrak{M}, \mu)$ be a positive measure space. Let $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $L^p(\mu)$. Prove the following conditions are equivalent:
(1) For any $f \in L^q(\mu)$, we have

$$\sum_{n=1}^{\infty} \left| \int_X g_n(x) f(x) \, d\mu(x) \right| < \infty.$$

(2) There exists $M \in [0, \infty)$ such that for any sequence $(\varepsilon_k)_{k=1}^{n}$ in $\mathbb{C}$ with $|\varepsilon_k| = 1$ for $k = 1, 2, \ldots, n$, we have

$$\left\| \sum_{k=1}^{n} \varepsilon_k g_k(x) \right\|_p \leq M.$$

5. Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of normed vector spaces. Define E to be the normed vector space of all sequences $(\xi_1, \xi_2, \ldots)$ such that $\xi_n \in E_n$ for all $n \in \mathbb{Z}_{>0}$ and such that $\lim_{n\to\infty} \|x_n\| = 0$, with the norm

$$\|(\xi_1, \xi_2, \ldots)\| = \sup_{n\in\mathbb{Z}_{>0}} \|\xi_n\|.$$

In the following, you do not need to prove that E is a vector space or that the formula above defines a norm on E.
(1) Suppose that E_n is complete for all $n \in \mathbb{Z}_{>0}$. Prove that E is a Banach space.
(2) Suppose that E_n is separable for all $n \in \mathbb{Z}_{>0}$. Prove or disprove: E is separable.

6. Define a function $f\colon [0,1] \to \mathbb{C}$ by

$$f(x) = \begin{cases} x^{3/2} \sin\left(\frac{1}{x}\right) & x \in (0,1] \\ 0 & x = 0. \end{cases}$$

Prove that f is absolutely continuous on $[0,1]$.

7. Set $\Omega = \{z \in \mathbb{C}\colon \mathrm{Re}(z) > -2\}$. Let f be a bounded holomorphic function on Ω such that $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right)$ for all $n \in \mathbb{Z}_{>0}$. Prove that f is constant.

8. Let $\Omega \subset \mathbb{C}$ be an open set, let $a \in \Omega$, and let f be a holomorphic function on $\Omega \setminus \{a\}$ which does not vanish on $\Omega \setminus \{a\}$. Prove that f has an essential singularity at a if and only if the function $z \mapsto \frac{1}{f(z)}$ has an essential singularity at a.

9. Let f be a holomorphic function on $\{z \in \mathbb{C}\colon |z| < 2\}$. Prove that

$$\sup_{|z|=1} |f(z) - \overline{z}| > \tfrac{1}{2}.$$