ANALYSIS QUALIFYING EXAM, FALL 2013

1. Let $(X, \mathfrak{M}, \mu)$ be a positive measure space and let $(E_n)_{n\in\mathbb{N}}$ be a sequence of measurable sets in $\mathfrak{M}$. For $n \in \mathbb{N}$, let $f_n = \chi_{E_n}$ be the characteristic function of E_n.

(1) Prove that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to 1 on X if and only if there exists $N \in \mathbb{N}$ such that $E_n = X$ for all $n > N$.

(2) Prove that $(f_n)_{n\in\mathbb{N}}$ converges almost everywhere to 1 on X if and only if

$$\mu\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}(X \setminus E_k)\right) = 0.$$

2. Let $\mathfrak{L}$ be the collection of Lebesgue measurable subsets of $[0,1]$, and let m be Lebesgue measure on $\mathfrak{L}$. Let $f\colon [0,1] \to \mathbb{C}$ be a measurable function. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1(m)$ such that:

(1) $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in [0,1]$ with respect to m.

(2) For every $\varepsilon > 0$ there is $\delta > 0$ such that for any $E \in \mathfrak{L}$ with $m(E) < \delta$ and every $n \in \mathbb{N}$, we have $\left|\int_E f_n \, dm\right| < \varepsilon$.

Prove that $f \in L^1(m)$ and that $\lim_{n\to\infty} \int_X |f_n - f| \, dm = 0$.

3. Let m be Lebesgue measure on $[0,1]$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of Lebesgue measurable functions on $[0,1]$. Suppose that there exists $f \in L^1([0,1])$ such that $|f_n(x)| \leq f(x)$ for all $x \in [0,1]$. For $n \in \mathbb{Z}_{>0}$, define a function $g_n \in C([0,1])$ by $g_n(x) = \int_0^x f_n \, dm$. Prove that the closure of $\{g_n : n \in \mathbb{Z}_{>0}\}$ in $C([0,1])$ is a compact subset of $C([0,1])$.

4. Let E be an inner product space. We say that a sequence $(\xi_n)_{n\in\mathbb{N}}$ in E is a *weak Cauchy sequence* if for any $\eta \in E$, the sequence $(\langle \eta, \xi_n\rangle)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$. We say that E is *sequentially weakly complete* if for any weak Cauchy sequence $(\xi_n)_{n\in\mathbb{N}}$ in E there is $\xi \in E$ such that for any $\eta \in E$,

$$\lim_{n\to\infty} \langle \eta, \xi_n \rangle = \langle \eta, \xi \rangle.$$

Suppose that E is complete. Prove that E is sequentially weakly complete.

5. Suppose X is a locally compact metric space and $T\colon C_0(X) \to C_0(X)$ is a bounded linear operator. Show that the range of T is not dense in $C_0(X)$ if and only if there exists a complex measure $\mu \neq 0$ on X such that for any $f \in C_c(X)$,

$$\int_X (Tf) \, d\mu = 0.$$

6. Set $D = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq x\}$. Let m_2 be Lebesgue measure on $\mathbb{R}^2$. Compute

$$\int_D x^{-3/2} e^{y-x} \, dm_2(x,y).$$

(Hint: You may use the fact that $\int_0^\infty x^{-1/2} e^{-x} \, dx = \pi^{1/2}$.)

7. Let $\Omega \subset \mathbb{C}$ be open, and let f be a holomorphic function on Ω. Let $A \subset \Omega$ be a subset with no cluster points in Ω. Suppose that $f|_{\Omega \setminus A}$ is injective. Prove that f is injective.

8. Set $D = \{z \in \mathbb{C} \colon |z| < 1\}$. Let $C(\overline{D})$ be, as usual, the Banach space of all continuous complex valued functions on $\overline{D}$ with the supremum norm $\| \cdot \|_{\infty}$. Let $A \subset C(\overline{D})$ be the subspace

$$A = \{f \in C(\overline{D}) \colon f|_D \text{ is holomorphic}\}.$$

Prove that there is a continuous linear functional $\omega \colon C(\overline{D}) \to \mathbb{C}$ such that $\omega(f) = f''\left(\frac{2}{3}\right)$ for all $f \in A$.

9. Let f be an entire function such that $\left|f\left(\frac{1}{n}\right)\right| \leq n^{-n}$ for all $n \in \mathbb{Z}_{>0}$. Prove that f is constant.