ON THE DERIVATION MODULE AND APOLAR ALGEBRA OF AN ARRANGEMENT OF HYPERPLANES

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In this dissertation we study the module of derivations and apolar algebra defined by an arrangement of hyperplanes. An arrangement of hyperplanes, $\mathcal{A}$ is a finite set of codimension 1 linear spaces in a finite dimensional vector space. Each hyperplane, H, can be defined by the kernel of a linear functional, call it α_H.

The module of derivations, $D(\mathcal{A})$, is the submodule of all polynomial derivations, θ, such that for all $H \in \mathcal{A}$ we have $\theta(\alpha_H) \in (\alpha_H)$. A preeminent problem in the field of arrangements of hyperplanes is to detect when $D(\mathcal{A})$ is a free module. M. Yoshinaga has shown that the freeness of $D(\mathcal{A})$ can be exposed through the exponents of the restricted multiarrangements. Using this as motivation we study exponents of multiarrangements in $\mathbb{CP}^1$. As a corollary we prove the Terao conjecture for arrangements in dimension three of size less than 11.

An apolar algebra $A(p)$ of a polynomial p is the quotient of the polynomial ring of differential operators with constant coefficients by the annihilator ideal of p. In this thesis we restrict to the case where p is a product of linear forms with integer exponents (i.e. p is the defining polynomial of a arrangement or multiarrangement). The only cases of arrangements whose apolar algebra, $A(p)$, is a complete intersection algebra previously known were reflection arrangements. In this thesis we study this algebra, $A(p)$, and construct examples of arrangements whose apolar algebra is a complete intersection that are not reflection arrangements and are not free. We also show the complete intersection property is not combinatorial.

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DEDICATION

To my parents for their unending support and love.

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CHAPTER I

INTRODUCTION

I.1. The Beginning

One of the most celebrated themes in mathematics is the strength of the connection between its different fields. A subject that exemplifies these interactions of disparate fields is the study of hyperplane arrangements. An arrangement of hyperplanes is a finite collection of codimension one linear spaces. It seems hard to find a true origin of the subject, but there are a few foundational papers that are the first to mention an arrangement of hyperplanes.

Maybe the first paper directly on the subject of hyperplane arrangements was published by Proceedings of the London Mathematical Society titled "On the figures formed by the intercepts of a system of straight lines in a plane, and on analogous relations in space of three dimensions" by S. Roberts in 1889 (see [15]). Then almost a century later, in 1971 a foundational paper by Grünbaum titled "Arrangements of hyperplanes" summarized what was known about the subject and predicted that the subject was to become more popular in the future (see [8]). Around that same time K. Saito spurred the study of a free hypersurface through the inspiration of singularity theory (see [16]). K. Saito's student Hiroaki Terao restricted Saito's definition to the case where the hypersurface is an arrangement of hyperplanes. In 1981 Terao published a beautiful formula relating the exponents of a free arrangement to its intersection lattice (see [19], [20], and [21]). This led Terao to conjecture that the freeness of a arrangement depends only on the intersection lattice. The conjecture is still open after more than 20 years and is the main motivation for this thesis.

In this thesis free hyperplane arrangements in three dimensions are studied along with the apolar algebra of an arrangement. To study free arrangements in three dimensions we use a recent result of Yoshinaga (also a student of Saito) that focusses on the exponents of a restricted multiarrangement (see [22] and [23]). Thus, a main character of the thesis is multiarrangements in dimension two. Given that exponents are always well-defined for a two dimensional multiarrangement, in this case we prove formulas for the exponents. A result of this study is a corollary that proves Terao's conjecture for certain large classes of arrangements.

Next, in the last chapter, we study the apolar algebra of an arrangement of hyperplanes. Many of the known examples of free arrangements have complete intersection apolar algebras. Thus, we are interested in finding conditions on when the apolar algebra is a complete intersection. We build a foundational lemma to test whether or not an apolar algebra is a complete intersection. We also use catalecticant determinants to study apolar algebras. Then we exhibit some of the first known examples of arrangements whose apolar algebras are complete intersections, but the arrangement is not a subarrangement of a reflection arrangement. These examples are not free, hence we show the property that the apolar is a complete intersection does not imply the arrangement is free. These examples are built from finite reflection groups and hence are fragile to perturbation of one hyperplane. We perturb one hyperplane of one example where the apolar algebra is a complete intersection so that the combinatorics of the intersection lattice is maintained. It turns out that this perturbed arrangement's apolar algebra is not a complete intersection. Thus, we have shown that the complete intersection property of an arrangement is not combinatorial (i.e. this is a counter example to the complete intersection version of Terao's conjecture).

I.2. Notation and basic facts

For the entirety of the thesis all the calculations will be done over the complex numbers, even though many of the results can be stated more generally. The main reference for nearly all the material concerning arrangements of hyperplane used here is the classical book by P. Orlik and H. Terao (see [14]). Let V be a finite dimensional vector space over $\mathbb{C}$ of dimension ℓ. Denote by $S = \mathcal{S}(V*)$ the symmetric algebra and choose coordinates such that $S = \mathbb{C}[x_1, \ldots, x_\ell]$. For simplicity let $\partial_i = \dfrac{\partial}{\partial_{x_i}}$ be the derivative with respect to x_i. By a hyperplane we mean a dimension $\ell - 1$ affine space $H \subset V$. Any hyperplane H is the kernel of a degree one form in S and denote this linear form by α_H (determined up to a constant). With this we have $H = \{v \in V \mid \alpha_H(v) = 0\}$. Then an arrangement of hyperplanes is a finite collection of hyperplanes, say n of them, in V. We will denote an arrangement by

$\mathcal{A} = \{H_1, \ldots, H_n\}$. Since each hyperplane is defined, up to a non-zero scalar, by a linear form we call the defining polynomial of an arrangement

$$Q_{\mathcal{A}} = \prod_{H \in \mathcal{A}} \alpha_H$$

and it is unique up to a scalar multiple. Also, given a polynomial of this form, a product of linear forms, we have defined an arrangement. An arrangement is called central when all the hyperplanes contain the origin of V. Further, an arrangement is called essential when the intersection of all the hyperplanes in $\mathcal{A}$ is the origin in V.

Notice that we can always "projectivize" a central arrangement $\mathcal{A} \subset \mathbb{C}^\ell$ and view the arrangement as codimension one linear spaces in $\mathbb{CP}^{\ell-1}$. For example, a hyperplane arrangement where $\ell = 3$ is an arrangement of planes in $\mathbb{C}^3$ and this setting is equivalent to studying the corresponding arrangement of lines in $\mathbb{CP}^2$. In this thesis we will often use this identification. Also, whenever we say "in general position" we mean in "some" nonempty open set in the Zariski topology.

One of the most important invariants of an arrangement is its intersection lattice; denote the intersection lattice of an arrangement $\mathcal{A}$ by $L(\mathcal{A})$. The points of this lattice are given by all possible intersections of hyperplanes in $\mathcal{A}$. Then the ordering is given by reverse inclusion. So,

$$L(\mathcal{A}) = \left\{ X \subseteq V \mid \exists H_{i_1}, \ldots, H_{i_m} \in \mathcal{A} \text{ such that } \bigcap_{j=1}^{m} H_{i_j} = X \right\}$$

and for $X, Y \in L(\mathcal{A})$

$$X \leq Y \text{ if and only if } X \supseteq Y.$$

This lattice is an important invariant of the arrangement and has nice properties. In particular, if $\mathcal{A}$ is a central arrangement then $L(\mathcal{A})$ is ranked by

$$\mathrm{rk}(X) = \mathrm{codim}(X).$$

Next, we define the Poincaré polynomial of the arrangement. To do this we first need the Möbius function $\mu : L(\mathcal{A}) \times L(\mathcal{A}) \to \mathbb{Z}$ which is defined recursively by $\mu(X,Y) = 0$ if $X \nleq Y$, $\mu(X,Y) = 1$ if $X = Y$, and otherwise

$$\sum_{X \leq Z \leq Y} \mu(X,Z) = 0.$$

Now, the Poincaré polynomial is an invariant of the arrangement and is coarser than the intersection lattice (see [14], Example 2.61).

Definition I.2.1. *The Poincaré polynomial of $\mathcal{A}$ is*

$$\pi(\mathcal{A},t) = \sum_{X \in L(\mathcal{A})} \mu(V,X) t^{rk(X)}.$$

This polynomial is the epicenter of the interaction of some disparate fields of mathematics. For example, it turns out that $\pi(\mathcal{A},t)$ is also the Hilbert series of the cohomology algebra of the complement of the arrangement $\mathcal{A}$ (this is the reason we call it the Poincaré polynomial). For another example see Terao's Factorization Theorem I.3.8.

I.3. Module of derivations

We study two algebraic objects attached to an arrangement of hyperplanes of interest in this thesis. The first is the module of derivations.

Definition I.3.1. *Let* $\mathrm{Der}_{\mathbb{C}}(S,S) = \{\theta : S \to S \mid \forall p, q \in S,\ \theta(pq) = p\theta(q) + q\theta(p)\}$.

It is clear that after choosing coordinates

$$\mathrm{Der}_{\mathbb{C}}(S,S) \cong S\partial_1 \oplus S\partial_2 \oplus \cdots \oplus S\partial_\ell.$$

Thus, for any derivation θ there exists unique polynomials f_i such that $\theta = \sum_{i=1}^{n} f_i \partial_i$.

Definition I.3.2. *The module of derivations of an arrangement of hyperplanes is the set of all derivations that preserve the defining polynomial of the arrangement:*

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}_{\mathbb{C}}(S,S) \mid \theta(Q_{\mathcal{A}}) \in SQ_{\mathcal{A}}\}.$$

Remark I.3.3. $D(\mathcal{A})$ *is a module over* S *by multiplication. However, it can be viewed geometrically as the set of all polynomial vector fields that are tangent to the arrangement.*

Since $D(\mathcal{A})$ is a sub-module of a free module it is a natural question to ask when this module is free. A partial answer is that it is rarely free (see [14], Example 4.34). So, we make the following definition.

Definition I.3.4. *An arrangement is free if the module of derivations* $D(\mathcal{A})$ *is a free module.*

There are only three main infinite families of examples of arrangements known that are free: 1) reflection arrangements, 2) supersolvable/fiber type arrangements, and 3) arrangements that satisfy an inductive criterion called recursive or addition/deletion arrangements. In particular, it is not known whether or not all free arrangements are recursively free (see [14], pp.122).

If $D(\mathcal{A})$ is free then we can choose a basis, $\{\theta_1, \ldots, \theta_\ell\}$, and write the module as $D(\mathcal{A}) = \langle \theta_1, \ldots, \theta_\ell \rangle_S$. This basis is not invariant over S but the degrees of the θ_i are invariant when we order them in increasing order. This leads us to the following definition.

Definition I.3.5. *The exponents of a free arrangement $\mathcal{A}$ are the degrees of the generators of its module of derivations in increasing order, and we write*

$$\exp(\mathcal{A}) = (\deg(\theta_1), \ldots, \deg(\theta_\ell)).$$

We will usually denote these exponents by $\exp(\mathcal{A}) = (e_1, e_2, \ldots, e_\ell)$. These exponents have turned out to be an important invariant of the arrangement. To help calculate a basis for this module and hence find an arrangement's exponents the following theorem due to Saito is useful.

Theorem I.3.6 (Saito's Criterion, [16])**.** *If the derivations $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A})$ are linearly independent over S then $\{\theta_1, \ldots, \theta_\ell\}$ is a basis for $D(\mathcal{A})$ if and only if*

$$\sum_{i=1}^{\ell} \deg(\theta_i) = \deg(Q_{\mathcal{A}}) = n.$$

Example I.3.7. *Let* $\ell = 3$ *and let* $\mathcal{A}$ *be defined by* $Q(\mathcal{A}) = x_1x_2x_3$*. Then* $\mathcal{A}$ *is a free arrangement (called the Boolean arrangement in dimension 3) and* $D(\mathcal{A})$ *is generated by* $\theta_1 = x_1\partial_1$*,* $\theta_2 = x_2\partial_2$*, and* $\theta_3 = x_3\partial_3$*. These three derivations are linearly independent and their degrees add up to 3. Thus,* $\exp(\mathcal{A}) = (1, 1, 1)$*. Notice that* $\pi(\mathcal{A}, t) = (1 + t)^3$*.*

In 1981, Hiroaki Terao published the following beautiful theorem relating the intersection lattice of a free arrangement $\mathcal{A}$ to its module of derivations (see [19], [20], and [21]). The theorem shows that the Poincaré polynomial of a free arrangement actually factors with the roots being determined by the lattice.

Theorem I.3.8 (Factorization)**.** *If an arrangement* $\mathcal{A}$ *is free with the following exponents* $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$ *then*

$$\pi(\mathcal{A}, t) = \prod_{i=1}^{\ell}(1 + d_i t).$$

This theorem helped Terao to make the following conjecture.

Conjecture I.3.9 (Terao)**.** *The freeness of an arrangement* $\mathcal{A}$ *depends only on the intersection lattice* $L(\mathcal{A})$*.*

This conjecture is still unsolved. Another way to state the conjecture is to say that if two arrangements have the same intersection lattice and one of them is free then the other arrangement is also free. Thus, the moduli space of an arrangement, the set of all arrangements which have isomorphic intersection lattices, contains all the information needed for this conjecture. We say that Terao's conjecture holds for an arrangement $\mathcal{A}$ if it is true for the moduli space of $L(\mathcal{A})$. Denote this moduli space of an intersection lattice by $M(L(\mathcal{A}))$. The largest advance towards Terao's conjecture could arguably be the following theorem proved by S. Yuzvinsky in 1993.

Theorem I.3.10 (Yuzvinsky, [26]). *The subset of $M(L(\mathcal{A}))$ consisting of all free arrangements is an open set in the Zarisky topology.*

The next significant work towards understanding the module of derivations came in recent work by M. Yoshinaga (see [22] and [23]). He proved that along with the property that an arrangement's Poincaré polynomial factors if a restricted multiarrangements exponents match the proposed exponents given by the Poincaré polynomial then the arrangement is free. To state Yoshinaga's theorem we need to slightly generalize an arrangement and its derivation module $D(\mathcal{A})$.

Definition I.3.11. *A multiarrangement of hyperplanes, denote it by $\tilde{\mathcal{A}}$, is an arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ together with positive integer multiplicities attached to each hyperplane. Denote the multiplicity of H_i by m_i and call $[m_1, \ldots, m_n]$ the multiplicity vector of $\tilde{\mathcal{A}}$.*

Ziegler was the first to study these objects. He defined the module of derivations of a multiarrangement.

Definition I.3.12. *The module of derivations of a multiarrangement $\tilde{\mathcal{A}} = \{H_1, \ldots H_n\}$*

with corresponding multiplicities $[m_1, \ldots, m_n]$ *is*

$$D(\tilde{\mathcal{A}}) = \left\{ \theta \in \mathrm{Der}_{\mathbb{C}}(S, S) \mid \forall H_i \in \tilde{\mathcal{A}},\ \theta(\alpha_i) \in (\alpha_i)^{m_i} S \right\}.$$

We can similarly define exponents of a free multiarrangement and write them as $\exp(\tilde{\mathcal{A}}) = (e_1, \ldots, e_\ell)$. Ziegler generalized Saito's criterion to the following.

Theorem I.3.13 (Ziegler, [27]). *If* $\theta_1, \theta_2, \ldots, \theta_\ell \in D(\tilde{\mathcal{A}})$ *are linearly independent over* S *then* $\{\theta_1, \theta_2, \ldots, \theta_\ell\}$ *is a basis for* $D(\tilde{\mathcal{A}})$ *if and only if*

$$\sum_{i=1}^{\ell} \deg(\theta_i) = \sum_{j=1}^{n} m_j.$$

For this thesis we concentrate on line arrangements in $\mathbb{CP}^2$. For Yoshinaga's theorem we will need multiarrangements in dimension two. Notice that in dimension two the module $D(\tilde{\mathcal{A}})$ is of rank two. We also know that for any dimension the module $D(\tilde{\mathcal{A}})$ is reflexive (see [27]). Thus, if $\ell = 2$ then $D(\tilde{\mathcal{A}})$ is always free. This proves that for any two dimensional multiarrangement the exponents are always well-defined. A multiarrangement in dimension two is equivalent to a multiarrangement of points in $\mathbb{CP}^1$. Note we can regard a multiarrangement of points in $\mathbb{CP}^1$ as an effective divisor in $\mathbb{CP}^1$. Each arrangement defines many multiarrangements via restriction.

Definition I.3.14. *If we have an arrangement* $\mathcal{A} = \{H_1, \ldots, H_n\}$ *in dimension* ℓ *then there are* n *multiarrangements defined by restriction. They are in dimension* $\ell - 1$ *and given by*

$$\tilde{\mathcal{A}}^i = \{H_j \cap H_i \mid j \neq i\}$$

and the natural multiplicity is given by

$$m(H_j \cap H_i) = |\{k \mid k \neq i \text{ and } H_k \cap H_i = H_j \cap H_i\}|.$$

Essentially, Yoshinaga has proved that these restriction arrangements can determine the freeness of the parent arrangement. Finally, we can state the relevant part of Yoshinaga's theorem for our purposes.

Theorem I.3.15 (Yoshinaga, [23])**.** *If an arrangement $\mathcal{A}$ of lines in $\mathbb{CP}^2$ is free with exponents $1, e_1, e_2$ then its restriction to every $\ell \in \mathcal{A}$ has (as a multi-arrangement) exponents e_1, e_2. If the Poincaré polynomial of $\mathcal{A}$ factors as*

$$\pi(\mathcal{A}, t) = (1+t)(1+e_1t)(1+e_2t)$$

and the restriction of $\mathcal{A}$ to some line has exponents e_1, e_2 then $\mathcal{A}$ is free with exponents $(1, e_1, e_2)$.

Yoshinaga's theorem implies in particular that if an arrangement $\mathcal{A}$ in $\mathbb{CP}^2$ gives a counterexample to Terao's conjecture then its restrictions to every line has exponents that are distinct from the coefficients in the factors of the Poincaré polynomial. Since the multiplicity vectors of the restrictions are determined by the intersection lattice of $\mathcal{A}$, if the exponents of any restricted multiarrangement were independent of the position of the points then Terao's conjecture holds for $\mathcal{A}$. In the case of arrangements with $\ell = 2$ the exponents are always $(1, n-1)$ because the Euler derivation is in every module of derivations (when all the multiplicities are 1). However, Ziegler was the first to notice that contrary to the case of arrangements, the exponents of a multiarrangement with $\ell = 2$ definitely depend on the position of the points (see Example II.2.5). This has provided the main motivation for this topic, of multiarrangements of points in $\mathbb{CP}^1$, in the thesis.

In Chapter II we calculate formulas for the exponents of some multiarrangements of points in $\mathbb{CP}^1$. We show that for certain multiplicity vectors the exponents do not depend on the position of the points. Then for multiplicity vectors whose exponents depend on the position of the points we prove the following main theorem of Chapter II.

Theorem I.3.16. *Let $\tilde{\mathcal{A}}$ be a multiarrangement of points in $\mathbb{CP}^1$ with multiplicity vector $[m_1, \ldots, m_n]$. If the points of $\tilde{\mathcal{A}}$ are in general position and $m_1 < \sum_{i=2}^{n} m_i$ and $\sum_{i=1}^{n} m_i \geq 2n - 1$ then*

$$\exp(\tilde{\mathcal{A}}) = \left(\left\lfloor \frac{1}{2} \sum_{i=1}^{n} m_i \right\rfloor, \left\lceil \frac{1}{2} \sum_{i=1}^{n} m_i \right\rceil \right)$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ are the floor and ceiling functions of the real number x respectively.

We prove this theorem by constructing a matrix defined by a multiplicity vector and showing that the theorem is true if and only if the determinant of this matrix is not zero. Then we show that the determinant is not zero. This determinant is a mysterious polynomial and we study some examples in Section II.8. Finally, we conclude Chapter II with Section II.9 and prove the following theorem.

Theorem I.3.17. *Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{CP}^2$ (equivalently an arrangement of hyperplanes in $\mathbb{C}^3$). If $|\mathcal{A}| < 11$ then Terao's conjecture holds for $\mathcal{A}$.*

I.4. Apolar algebras

The second algebraic object studied in this thesis defined by an arrangement of hyperplanes is the apolar algebra. Apolar forms and apolar algebras first appeared prolifically in the study of Waring's problem (see [4], [10], and [18]). Waring's problem is to find the smallest integer, k, such that a generic degree n polynomial can be expressed as the sum k linear forms raised to the exponent n. Waring's problem has been solved with the help of catalecticant determinants and Artinian Gorenstein algebras (see [1] and [10]). The solution used many different techniques from algebraic geometry (e.g. the Hilbert scheme and Gröbner bases). Thus, along the way to the solution many other peripheral question were raised.

Macaulay was the first to consider the annihilator ideal and the quotient algebra. He worked mainly with inverse systems (see [12]). He proved that all zero dimensional Gorenstein rings are apolar algebras along with the converse. He did this even though at the time there was not a notion of Gorenstein rings. He raised many questions about these algebras and these algebras have been found to have many connections to other problems in algebraic geometry and commutative algebra (see [2]).

One problem stumbled upon by algebraic geometers and commutative algebraists when studying Waring's problem was when is the apolar algebra of a polynomial a complete intersection. This problem is easy if the polynomial is generic; all apolar algebras of generic polynomials of degree greater than three are not complete intersections (see [10]). However, no conditions are known on a non-generic polynomial that imply its apolar algebra is a complete intersection. In Chapter III we study this question and restrict to the non-generic case where the polynomial is product of linear forms (i.e. an arrangement of hyperplanes).

In Chapter III we prove a lemma that helps us detect when an apolar algebra is a complete intersection. Then we study the apolar algebra of a generic hyperplane arrangement and compute the catalecticant determinants for generic arrangements of size 4 and 6. Next, we study reflection arrangements and a certain generalization of reflection arrangements. We find examples of arrangements whose apolar algebra is a complete intersection, but the arrangement is not free. The same examples allow us to construct another example by perturbation of a special hyperplane that proves a complete intersection version of Terao's conjecture (i.e. the property that an apolar algebra is a complete intersection is a combinatorial property) is false.

CHAPTER II

DERIVATION MODULES

II.1. Notation and preliminaries

In this chapter we concentrate on multiarrangements in $\mathbb{CP}^1$. This chapter contains new results concerning the derivation module of these multiarrangements and related results. A multiarrangement in $\mathbb{CP}^1$ is a finite collection of points, defined by $\tilde{\mathcal{A}} = \{H_1, \ldots, H_n\}$, with multiplicities, denoted by a multiplicity function $m : \tilde{\mathcal{A}} \to \mathbb{Z}_{>0}$ where for notational purposes we set $m(H_i) = m_i$. We also assume that $n \geq 2$ since the case when $n = 1$ is trivial. Each point H_i is defined by a linear functional α_i so that $\ker(\alpha_i) = H_i$. We can always choose coordinates so we can view the symmetric algebra $\mathcal{S}(\mathbb{CP}^1)$ as the polynomial algebra $S = \mathbb{C}[x, y]$ and that $\alpha_1 = x$ and $\alpha_2 = y$. This choice of coordinates defines numbers $p_3, p_4, \ldots, p_n \in \mathbb{C}$ such that for all $3 \leq i \leq n$ the defining forms are $\alpha_i = x - p_i y$ up to a nonzero scalar multiple. In Sections II.3, II.4, II.5, II.6, and II.7 we will treat the p_i as variables. We also can order the points in $\tilde{\mathcal{A}}$ such that $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_n$ and call $[m_1, m_2, \ldots, m_n]$ the multiplicity vector of $\tilde{\mathcal{A}}$. Denote the defining polynomials as $Q = \prod_{i=1}^{n} \alpha_i$ and

$\tilde{Q} = \prod_{i=1}^{n} \alpha_i^{m_i}$. With the above choice of coordinates we have that

$$\tilde{Q} = x^{m_1} y^{m_2} \prod_{i=3}^{n} (x - p_i y)^{m_i}.$$

For simplicity denote the partial derivatives by $\frac{\partial}{\partial x} = \partial_x$ and $\frac{\partial}{\partial y} = \partial_y$.

As described in the introduction, in this section we study the module of derivations of these multiarrangements:

$$D(\tilde{\mathcal{A}}) = \{\theta \in \mathrm{Der}_{\mathbb{C}}(S, S) \mid \forall i,\ \theta(\alpha_i) \in (\alpha_i^{m_i})S\}.$$

Recall, that we can write every derivation θ in the form $\theta = f\partial_x + g\partial_y$ for some polynomials f and g in the polynomial ring S. As stated in the introduction, this module is always free because it is reflexive and of rank 2. In this setting each multiarrangement has exponents defined as the degrees of homogeneous generators of $D(\tilde{\mathcal{A}})$. Denote these exponents as $\exp(\tilde{\mathcal{A}}) = (e_1, e_2)$ where $e_1 \leq e_2$. The overall goal of this section is to prove Theorem I.3.16: if the parameters p_i are generic and some conditions on the multiplicities are satisfied then

$$\exp(\tilde{\mathcal{A}}) = \left(\left\lfloor \frac{1}{2} \sum_{i=1}^{n} m_i \right\rfloor, \left\lceil \frac{1}{2} \sum_{i=1}^{n} m_i \right\rceil \right).$$

Another important part of this chapter is to define and study degeneration varieties. We conclude the chapter by proving Terao's conjecture for certain arrangements.

II.2. Combinatorial multiplicities

In this section we study cases where the exponents are determined by the multiplicities. So, in these cases the exponents are the same for any parameters with the fixed multiplicities. Recall that in this case where $\tilde{\mathcal{A}}$ is a multiarrangement in $\mathbb{CP}^1$ then $D(\tilde{\mathcal{A}})$ is always a free module. Thus, $D(\tilde{\mathcal{A}})$ always satisfies the generalized Saito's criterion (Theorem I.3.13). Satisfying the generalized Saito's criterion means that for any multiarrangement in $\mathbb{CP}^2$ we have

$$e_1 + e_2 = \sum_{i=1}^{n} m_i.$$

We focus on finding the exponents when the multiplicities are fixed. If the multiplicities are fixed then it is sufficient to find a minimal exponent e_1 (or equivalently find a generator of minimal degree) and then use the equality of the generalized Saito's criterion to find the other exponent e_2. In the next few lemmas we utilize this idea.

Lemma II.2.1. *If* $m_1 \geq \sum_{i=2}^{n} m_i$ *then* $(e_1, e_2) = \left(\sum_{i=2}^{n} m_i, m_1\right)$.

Proof. Consider the derivation $\theta = \frac{\tilde{Q}}{x^{m_1}}\partial_y$. For any $\tilde{\mathcal{A}}$ this derivation θ is in the module of derivations $D(\tilde{\mathcal{A}})$ since $\theta(x) = 0$ and

$$\theta(x - p_i y) = -p_i \frac{\tilde{Q}}{x^{m_1}} = -p_i \frac{\tilde{Q}}{x^{m_1}(x - p_i y)^{m_i}}(x - p_i y)^{m_i}$$

are in the ideals $(x^{m_1})S$ and $((x - p_i y)^{m_i})S$ respectively. Any homogeneous derivation ϕ which has a nonzero ∂_x term must be of the form $\phi = f x^{m_1}\partial_x + g\partial_y$. Notice that

$\deg(\phi) \geq m_1 \geq \sum_{i=2}^{n} m_i = \deg(\theta)$. Since $m_1 \geq \sum_{i=2}^{n} m_i$ any derivation in $D(\tilde{\mathcal{A}})$ that has polynomial degree less than or equal to m_1 must have no ∂_x term (i.e. that $f = 0$), other wise the coefficient x^{m_1} would be needed and that would raise the degree. Further, it is clear that θ is the smallest degree derivation that has no ∂_x term. This gives that θ is a minimal degree generator of $D(\tilde{\mathcal{A}})$. Thus, $e_1 = \sum_{i=2}^{n} m_i$ and $e_2 = m_1$. □

Lemma II.2.2. *If* $\sum_{i=1}^{n} m_i \leq 2n - 2$ *and* $m_1 < \sum_{i=2}^{n} m_i$ *then*

$$(e_1, e_2) = \left(\sum_{i=1}^{n} m_i - n + 1, n - 1 \right).$$

Proof. Since $m_1 < \sum_{i=2}^{n} m_i$ then the derivation θ from above in Lemma II.2.1 will have degree greater than half the sum of the multiplicities. However, since $\sum_{i=1}^{n} m_i \leq 2n-2$ we find a different derivation that does have degree less than half the sum of the multiplicities. The derivation we consider is $\bar{\theta} = \frac{\tilde{Q}}{Q}\theta_E = \frac{\tilde{Q}}{Q}(x\partial_x + y\partial_y)$. Now, $\bar{\theta}$ is in $D(\tilde{\mathcal{A}})$ since the Euler derivation $\theta_E = x\partial_x + y\partial_y$ has the property $\theta_E(\alpha_i) = \alpha_i$. Notice $\deg(\bar{\theta}) = \sum_{i=1}^{n} m_i - n + 1$. Since $\sum_{i=1}^{n} m_i \leq 2n - 2$ it follows that we have $\deg(\bar{\theta}) = \sum_{i=1}^{n} m_i - n + 1 \leq 2n - 2 - n + 1 = n - 1$. Now, we want to show that $\bar{\theta}$ is in the basis of $D(\tilde{\mathcal{A}})$. Assuming θ_1 is the minimal generator and that $\deg(\bar{\theta}) \leq e_2$ we know that $\bar{\theta} = f\theta_1$ for some polynomial f. If this is true then $\theta_1 = \frac{1}{f}\bar{\theta}$ and f divides $\frac{\tilde{Q}}{Q}$. If f is a polynomial then either it is some product of the defining linear functionals α_i or it is just a constant. If it was a product of the α_i's then θ_1 could not be in $D(\tilde{\mathcal{A}})$ because, say α_i was a factor of f, then $\theta_1(\alpha_i) \notin (\alpha_i^{m_i})$ since there would be a missing α_i. This gives that f is a non-zero constant. Thus, $\bar{\theta}$ is a minimal in degree generator and $e_1 = \sum_{i=1}^{n} m_i - n + 1$. By the generalized Saito's criterion $e_2 = n - 1$. □

Lemma II.2.3. *If* $\sum_{i=1}^{n} m_i = 2n - 1$ *and* $m_1 < \sum_{i=2}^{n} m_i$ *then* $(e_1, e_2) = (n-1, n)$.

Proof. Suppose that there is a an arrangement $\tilde{\mathcal{A}}$ with the above multiplicities and a derivation $\tilde{\theta} \in D(\tilde{\mathcal{A}})$ such that $\deg(\tilde{\theta}) = n - 2$. Then consider the arrangement, call it $\tilde{\mathcal{A}} - 1$, obtained by deleting one multiplicity from the last hyperplane. This arrangement will have multiplicities $[m_1, \ldots, m_{n-1}, m_n - 1]$. These multiplicities fit into the criterion of Lemma II.2.2 so that the exponents must be $\left(\sum_{i=1}^{n} m_i - n + 1, n - 1\right) = (n-1, n-1)$. Since $\tilde{\theta} \in D(\tilde{\mathcal{A}})$ then we must have $\tilde{\theta} \in D(\tilde{\mathcal{A}} - 1)$ but this is a contradiction because we assumed the degree of $\tilde{\theta}$ is $n - 2$ and there are not any derivations in $D(\tilde{\mathcal{A}} - 1)$ of degree less than $n - 1$. Therefore, there can not be any derivations of degree less than $n - 1$, so the exponents must be $(n - 1, n)$. □

Lemma II.2.4. *If* $m_i = 2$ *for all* $1 \leq i \leq n$ *then the exponents are* (n, n).

Proof. In this case we know that $\tilde{Q} = Q^2$. First, suppose that $n = 2$. Then $\tilde{Q} = x^2y^2$ so put $\theta_1 = x^2\partial_x$ and $\theta_2 = y^2\partial_y$. Then with the generalized Saito's criterion $\{\theta_1, \theta_2\}$ is a basis for $D(\tilde{\mathcal{A}})$ and $\exp(\tilde{\mathcal{A}}) = (2, 2)$. Now, suppose that $n > 2$ and for $2 < i \leq n$

define $Q_i := \dfrac{Q}{x - p_i y}$. Put $h_1 = \sum_{i=3}^{n} Q_i$ and $h_2 = \sum_{i=3}^{n} p_i Q_i$. Define the derivations

$$\theta_1 = h_1\theta_E + \frac{y}{x}Q\partial_y$$

and

$$\theta_2 = h_2\theta_E - \frac{x}{y}Q\partial_x.$$

Then we compute the matrix

$$M(\theta_1, \theta_2) = \begin{bmatrix} xh_1 & xh_2 - \frac{x}{y}Q \\ yh_1 + \frac{y}{x}Q & yh_2 \end{bmatrix}.$$

Then the determinant is

$$\det(M(\theta_1, \theta_2)) = xyh_1h_2 - (xyh_1h_2 + yh_2Q - xh_1Q - Q^2)$$

$$= Q^2 + Q(xh_1 - yh_2) = Q^2 + Q\left(\sum_{i=3}^{n} xQ_i - \sum_{i=3}^{n} p_i y Q_i\right)$$

$$= Q^2 + Q\left(\sum_{i=3}^{n}(x - p_i y)Q_i\right) = Q^2 + Q\left(\sum_{i=3}^{n} Q\right) = (n-1)Q^2.$$

Since the characteristic of the field is zero this determinant is not zero and satisfies the generalized Saito's criterion. Now we need to check that these derivations are in $D(\tilde{\mathcal{A}})$. We need to check for every α_i and $j = 1, 2$ we have $\theta_j(\alpha_i) \in (\alpha_i)^{m_i}S$. We know $\theta_1(x) = h_1 x$ and each term of h_1 is divisible by x. So, we have that $\theta_1(x) \in (x^2)S$. The analogous argument shows that $\theta_2(y) \in (y^2)S$, $\theta_1(y) \in (y^2)S$, and $\theta_2(x) \in (x^2)S$. Now, for all $3 \leq i \leq n$ we have that

$$\theta_1(x - p_i y) = h_1(x - p_i y) - p_i \frac{y}{x} Q$$

$$= \sum_{j=3, j\neq i}^{n} Q_j(x - p_i y) + Q/x(x - p_i y) \in (x - p_i y)^2 S.$$

By a similar calculation for all $3 \leq i \leq n$ we have $\theta_2(x - p_i y) \in (x - p_i y)^2 S$.

□

We remark here that this last lemma was originally done by Solomon and Terao in [17].

Since there is only one position of $n = 1, 2$, or 3 points in $\mathbb{CP}^1$ up to projective isomorphism we know the exponents are determined by the multiplicities (in this case the exponents are the floor and ceiling of half the sum of the multiplicities as in the main theorem). However, with 4 points there is more than one projective isomorphism type of configurations. In this case and cases with more points where the multiplicities do not satisfy the hypothesis of Lemma II.2.1, Lemma II.2.2, Lemma II.2.3, or Lemma II.2.4 the exponents can vary.

Example II.2.5 (Ziegler, [27])**.** *Define* $\tilde{\mathcal{A}}_1$ *by* $\tilde{Q}_1 = x^3y^3(x - y)(x + y)$ *and* $\tilde{\mathcal{A}}_2$ *by* $\tilde{Q}_2 = x^3y^3(x - y)(x + p_4 y)$ *where* $p_4 \neq 1$*. So, these multiarrangements have the same multiplicities* $[3, 3, 1, 1]$*. However, their exponents are different:*

$$\exp(\tilde{\mathcal{A}}_1) = (3, 5)$$

and

$$\exp(\tilde{\mathcal{A}}_2) = (4, 4).$$

We study this example in more depth in Example II.8.2.

In addition to showing that the exponents of a multiarrangement depend on the position of the points, this example shows that the results in Lemmas II.2.1, II.2.2, II.2.3, and II.2.4 are tight. However, at this time the author does not know whether or not these are the only cases of multiplicity vectors where the exponents are invariant under a change in the position of the points. This led the author to make the following conjecture.

Conjecture II.2.6. *The multiplicity vectors that satisfy Lemmas II.2.1, II.2.2, II.2.3, and II.2.4 are the only multiplicity vectors where the exponents do not depend on the position of the points.*

In the next few sections we will focus on multiarrangements in $\mathbb{CP}^1$ that are not combinatorially determined like that of Example II.2.5. The goal is to compute the exponents for points in general position.

II.3. The matrix of a non-combinatorial multiplicity vector

The goal of this section is to construct a matrix that we will use to prove the main theorem. Then in later sections we will prove that its determinant is not equal to zero. The following assumptions are important for the arguments below:

1. $\sum_{i=1}^{n} m_i$ is even,
2. for all i the multiplicities satisfy $m_i \geq 2$,

3. $m_1 < \sum_{i=2}^{n} m_i$, and

4. $e_1 = \left(\frac{1}{2}\sum_{i=1}^{n} m_i\right) - 1$.

Choose a homogeneous generator of $D(\tilde{\mathcal{A}})$, say θ_1, such that $\deg(\theta_1) = e_1$. Then θ_1 is of the form $\theta = x^{m_1} f(x,y)\partial_x - y^{m_2} g(x,y)\partial_y$ where $f(x,y), g(x,y) \in S$, the degree is $\deg(f(x,y)) = e_1 - m_1$, and $\deg(g(x,y)) = e_1 - m_2$. Since $f(x,y)$ and $g(x,y)$ are homogeneous polynomials of two variables, we write

$$f(x,y) = \sum_{j=0}^{e_1-m_1} f_j x^{e_1-m_1-j} y^j$$

and

$$g(x,y) = \sum_{j=0}^{e_1-m_2} g_i x^{e_1-m_2-j} y^j$$

where $f_j, g_j \in \mathbb{C}$. By definition $\theta_1 \in D(\tilde{\mathcal{A}})$ implies that for all $i \geq 3$ we have $\theta_1(x - p_i y) \in (x - p_i y)^{m_i} S$. Here, if we set the parameter y equal to one then we get that $\theta_1(x - p_i y)|_{y=1} \in (x - p_i)^{m_i}\mathbb{C}[x]$. Now, the setting is one variable where we can use college freshman calculus. We have $\theta_1(x - p_i y)|_{y=1} \in (x - p_i)^{m_i}\mathbb{C}[x]$ if and only if $(x-p_i)^{m_i}$ divides $\theta_1(x - p_i y)|_{y=1}$. Then $(x-p_i)^{m_i}$ divides $\theta_1(x-p_i y)|_{y=1}$ if and only if $(x-p_i)$ divides $\partial_x^k(\theta_1(x-p_i y)|_{y=1})$ for all $0 \leq k \leq m_i - 1$ where ∂_x^k is the k^{th} derivative with respect to x. Thus, $\theta_1 \in D(\tilde{\mathcal{A}})$ if and only if $\partial_x^k(\theta_1(x - p_i y)|_{y=1})|_{x=p_i} = 0$ for all $0 \leq k \leq m_i - 1$. Conveniently, we can calculate $\partial_x^k(\theta_1(x - p_i y)|_{y=1})|_{x=p_i}$. First, we calculate

$$(\theta_1(x - p_i y)|_{y=1})|_{x=p_i} = \sum_{j=0}^{e_1-m_1} f_j p_i^{e_1-j} + p_i \sum_{j=0}^{e_1-m_2} g_j p_i^{e_1-m_2-j}.$$

Calculation of the derivatives results in

$$\partial_x^k(\theta_1(x-p_iy)|_{y=1})|_{x=p_i}$$

$$=\sum_{j=0}^{e_1-m_1} f_j\frac{(e_1-j)!}{(e_1-j-k)!}p_i^{e_1-j-k}+p_i\sum_{j=0}^{e_1-m_2} g_j\frac{(e_1-m_2-j)!}{(e_1-m_2-j-k)!}p_i^{e_1-m_2-j-k}.$$

Now, we construct a matrix $\mathcal{M}$ with the coefficients of the f_j and the g_k. The matrix will consist of $n-2$ levels of which each has m_i rows and each entree in this level is a monomial in the variable p_i respectively, where $3\leq i\leq n$. Call these levels L_i. The st^{th} entree in the i^{th} level L_i is the s^{th} term without the coefficient f_j or g_j of the above expansion of $\partial_x^{t-1}(\theta_1(x-p_iy)|_{y=1})|_{x=p_i}$. So, the first half of the level L_i (given by f), call this first half FH_i, is the columns $1,2,\ldots,e_1-m_1+1$ given by

$$FH_i=\begin{bmatrix} p_i^{e_1} & p_i^{e_1-1} & \cdots & p_i^{m_1}\\ e_1p_i^{e_1-1} & \begin{bmatrix}e_1-1\\1\end{bmatrix}p_i^{e_1-2} & \cdots & m_1p_i^{m_1-1}\\ \begin{bmatrix}e_1\\2\end{bmatrix}p_i^{e_1-2} & \begin{bmatrix}e_1-1\\2\end{bmatrix}p_i^{e_1-3} & \cdots & \begin{bmatrix}m_1\\2\end{bmatrix}p_i^{m_1-2}\\ \vdots & \vdots & \ddots & \vdots\\ \begin{bmatrix}e_1\\e_1-k\end{bmatrix}p_i^{e_1-k} & \begin{bmatrix}e_1-1\\k\end{bmatrix}p_i^{e_1-1-k} & \cdots & \begin{bmatrix}m_1\\k\end{bmatrix}p_i^{m_1-k}\\ \vdots & \vdots & \ddots & \vdots\\ \begin{bmatrix}e_1\\m_i-1\end{bmatrix}p_i^{e_1-(m_i-1)} & \begin{bmatrix}e_1-1\\m_i-1\end{bmatrix}p_i^{e_1-m_i} & \cdots & \begin{bmatrix}m_1\\m_i-1\end{bmatrix}p_i^{m_1-(m_i-1)} \end{bmatrix}$$

where $\begin{bmatrix}n\\k\end{bmatrix}=k!\binom{n}{k}$. Then the second half of the level L_i (given by g), call this second half SH_i, is the columns $e_1 - m_1 + 2, e_1 + m_1 + 3, \ldots, \sum_{i=3}^{n} m_i$ given by

$$SH_i = \begin{bmatrix} p_i^{e_1-m_2+1} & p_i^{e_1-m_2} & \cdots & p_i^2 & p_i \\ \begin{bmatrix} e_1-m_2 \\ 1 \end{bmatrix} p_i^{e_2-m_2} & \begin{bmatrix} e_1-m_2-1 \\ 1 \end{bmatrix} p_i^{e_1-m_2-1} & \cdots & p_i & 0 \\ \begin{bmatrix} e_1-m_2 \\ 2 \end{bmatrix} p_i^{e_2-m_2-1} & \begin{bmatrix} e_1-m_2-1 \\ 2 \end{bmatrix} p_i^{e_1-m_2-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \begin{bmatrix} e_1-m_2 \\ k \end{bmatrix} p_i^{e_1-m_2+1-k} & \begin{bmatrix} e_1-m_2-1 \\ k \end{bmatrix} p_i^{e_1-m_2-k} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \begin{bmatrix} e_1-m_2 \\ m_i-1 \end{bmatrix} p_i^{e_1-m_2+1-(m_i-1)} & \begin{bmatrix} e_1-m_2-1 \\ m_i-1 \end{bmatrix} p_i^{e_1-m_2-m_i+1} & \cdots & 0 & 0 \end{bmatrix}.$$

This gives the i^{th} level by juxtaposition $L_i = FH_iSH_i$. Then the matrix is just these levels stacked on top of each other

$$\mathcal{M} = \begin{bmatrix} L_3 \\ \vdots \\ L_n \end{bmatrix}.$$

There are $e_1 - m_1 + 1$ columns in the first half of $\mathcal{M}$ coming from the polynomial f and $e_1 - m_2 + 1$ columns in the second half coming from the contribution of g. So, the total number of columns is $2e_1 - m_1 - m_2 + 2 = \sum_{i=3}^{n} m_i$. The total number of rows is also $\sum_{i=3}^{n} m_i$, where each m_i, $3 \leq i \leq n$, is given by the i^{th} level L_i. Note that $\mathcal{M}$ is independent from θ_1 and that we can treat the parameters p_i as variables so that the only information needed to construct $\mathcal{M}$ is the multiplicities $[m_1, \ldots, m_n]$. See

Section II.8 for examples of $\mathcal{M}$, in particular see Example II.8.2. In the next section we will study the determinant of $\mathcal{M}$, denote it by $P = \det(\mathcal{M})$. We will see that the main theorem is equivalent to proving this determinant is not zero.

II.4. The construction of the determinant of $\mathcal{M}$

Recall that we have made 4 important assumptions on the multiplicity vector:

1. $\sum_{i=1}^{n} m_i$ is even,
2. for all i we have $m_i \geq 2$,
3. $m_1 < \sum_{i=2}^{n} m_i$, and
4. $e_1 = \left(\frac{1}{2}\sum_{i=1}^{n} m_i\right) - 1$.

This is excluding the conditions of the lemmas of Section II.2. We need to assume this otherwise the determinant of the matrix $\mathcal{M}$ could be zero. If the first multiplicity is larger than the sum of the rest then the matrix will not be square since the first half, which is built from f, will not be defined. In Section II.7 we will consider the case where some of the multiplicities may be equal to 1 and show why the assumption that $\sum_{i=1}^{n} m_i \leq 2n - 1$ is important.

Let v be the vector in $\mathbb{C}^d$, where $d = \sum_{i=3}^{n} m_i$, with entries f_j and g_j, so that

$$v = \begin{bmatrix} f_0 \\ \vdots \\ f_{e_1 - m_1} \\ g_0 \\ \vdots \\ g_{e_1 - m_2} \end{bmatrix}.$$

If we have a multiarrangement $\tilde{\mathcal{A}}$ then we have a specific set of numbers assigned to the parameters $(p_3, \ldots, p_n)$. If the above θ_1 was in the module $D(\tilde{\mathcal{A}})$, which is one degree less than the proposed formula for the exponents, then v would be in the kernel of $\mathcal{M}$. However, we want to make a more general statement. So, treat $\mathcal{M}$ again as it was constructed with the parameters as variables. If we can show that the determinant of $\mathcal{M}$, $P = \det(\mathcal{M})$, is a non-zero polynomial in the variables p_i then $\mathcal{M}$ would only have a non-zero kernel for some non-generic set of numbers for the p_i. If $\mathcal{M}$ generically has a zero kernel then generically there can not be any derivations in $D(\tilde{\mathcal{A}})$ with lower degree than half the sum of the multiplicities, which proves the goal. So, we have restricted our attention to the calculation of this determinant P. Thus, the main theorem would follow from the following theorem.

Theorem II.4.1. *The polynomial P is not the zero polynomial.*

The determinant of $\mathcal{M}$ is a polynomial in the variables $p_3, \dots, p_n$. To show that this polynomial is non-zero we order the variables naturally, $p_3 > p_4 > \cdots > p_n$, and show the largest monomial of the determinant in lexicographic ordering (we denote the largest monomial in the lexicographic ordering of the polynomial P by $\mathtt{in}(P)$) is non-zero. To do this we use the Laplacian decomposition of the determinant by minors (a good reference is [13]). In each cofactor we find this largest monomial for a specific variable p_i. We choose the minors so that they fit exactly in the level L_i (i.e. they are size $m_i \times m_i$ and sit inside L_i) and vary left to right inside the level L_i. But first we prove the following useful lemma (the author learned this from Hiroaki Terao).

Lemma II.4.2. *Given the following smooth functions of one variable x:*

$$\{f_j(x) = x^{\lambda_j}\}_{j=1}^{k}$$

where $\lambda_1 > \cdots > \lambda_k \geq 0$ then the Wronskian of these functions is a monomial in x of degree $\sum_{j=1}^{k} \lambda_j - \sum_{r=1}^{k-1} r$ with coefficient

$$(-1)^{\lfloor \frac{k}{2} \rfloor} \prod_{1 \leq i < j \leq k} (\lambda_j - \lambda_i).$$

Proof. Call the matrix associated to the Wronskian M. First, we prove that the Wronskian is a monomial. We induct on k. Base case is that there is just one entry and we are done. We can compute $\det(M)$ by Laplaces decomposition in the first row (i.e. the cofactor expansion in the first row). So, $\det(M) = \sum_{j=1}^{k} a_{1j} A_j$ where A_i is the

j^{th} cofactor for the first row and a_{ij} is the ij^{th} entry. Now we compute $\deg(a_{1j}A_j)$ for any j. Examine the matrix of degrees of M:

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1 - 1 & \lambda_2 - 1 & \cdots & \lambda_k - 1 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 - k & \lambda_2 - k & \cdots & \lambda_k - k \end{bmatrix}.$$

Note that when the coefficient is zero we need not count that degree. Then we have $\deg(a_{1j}A_j) = \lambda_j + \deg(A_j)$ and the matrix of degrees of A_j is

$$\begin{bmatrix} \lambda_1 - 1 & \cdots & \lambda_{j-1} - 1 & \lambda_{j+1} - 1 & \cdots & \lambda_k - 1 \\ \lambda_2 - 2 & \cdots & \lambda_{j-1} - 2 & \lambda_{j+1} - 2 & \cdots & \lambda_k - 2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \lambda_2 - k & \cdots & \lambda_{j-1} - k & \lambda_{j+1} - k & \cdots & \lambda_k - k \end{bmatrix}.$$

Thus, by induction A_j is a monomial and

$$\deg(A_j) = \left(\sum_{i=1}^{k} [\lambda_i - 1]\right) - (\lambda_j - 1) - \sum_{i=1}^{k-2} i = \left(\sum_{i=1}^{k} \lambda_i\right) - \lambda_j - \sum_{i=1}^{k-1} i$$

which implies for any j

$$\deg(a_{1j}A_j) = \left(\sum_{i=1}^{k} \lambda_i\right) - \sum_{i=1}^{k-1} i.$$

Finally, since a_{1j} is a monomial for all $1 \leq j \leq k$ and the degrees of $a_{1j}A_j$ is the same for all j we have that $\det(M)$ is a monomial of degree desired.

Now, we show that the coefficient of the above monomial is the Vandermonde. We set the variable $x = 1$ and then use elementary row operations to get the matrix of coefficients of M to be exactly the Vandermonde. Since the matrix is the Wronskian of monomials with coefficient 1 we know that the second row, only considering the coefficients, is of the form $[\lambda_1\ \lambda_2\ \cdots \lambda_k]$. Then the next, third, row is of the form $[\lambda_1(\lambda_1 - 1) \cdots\cdots \lambda_k(\lambda_k - 1)] = \left[\lambda_1^2 - \lambda_1 \cdots\cdots \lambda_k^2 - \lambda_k\right]$. So, if we add the first row to the second we get $\left[\lambda_1^2 \cdots\cdots \lambda_k^2\right]$. This is the base case for induction on the number of rows. Now, the j^{th} row is of the form

$$\left[(j-1)!\binom{\lambda_1}{j-1} \cdots\cdots (j-1)!\binom{\lambda_k}{j-1}\right].$$

So, each entry in this row is a monic polynomial of degree one greater than the previous row. By induction we know that all the previous rows are in the form of the Vandermonde with each degree below the degree of the j^{th} rows entries. Thus, with elementary row operations we can eliminate all the lower terms in each of the entries on the j^{th} row. By induction we have exactly the Vandermonde matrix, except for the fact that it is upside down. So, we exchange rows i and $k - i + 1$ for all $1 \leq i \leq \lfloor\frac{k}{2}\rfloor$. Therefore, the coefficient of the monomial is

$$(-1)^{\lfloor\frac{k}{2}\rfloor} \prod_{1 \leq i < j \leq k} (\lambda_i - \lambda_j).$$

□

Now, we can use this lemma for a particular case of Theorem II.4.1. This case is where for each level L_i the degree of the monomials of the first half FH_i of $\mathcal{M}$ do not have the same degree of the entries in the second half SH_i.

Proposition II.4.3 (The case of no overlapping). *If $m_1 + m_2 > d$, again where $d = \sum_{i=3}^{n} m_i$, then the exponents of $\tilde{\mathcal{A}}$ are half the sum of the multiplicities.*

Proof. We have that $m_1 + m_2 > d$ if and only if $m_1 > e_1 - m_2 + 1$. Notice that the last statement is equivalent to the the last column of the first half of $\mathcal{M}$ having larger degree than the first column of the second half of $\mathcal{M}$, in the top row of each level. In this case there is no two columns that have the top entree in each level to be the same. We can enumerate the columns of the matrix $1, \ldots, \sum_{i=3}^{n} m_i$ left to right. Now the goal is to choose minors in the Laplacian decomposition of the determinant so that we get the highest degree term of the determinant. We do this essentially by starting at the top left and continuing down to the right. More precisely, the first minor is taken from the intersection of the all the rows of the first level L_3 and the columns $\{1, 2, \ldots, m_3\}$ and we continue in this same fashion so that the i^{th} minor consists of the intersection of all the rows of the $(i+2)^{th}$ level L_{i+2} and the columns

$$\left\{ \left(\sum_{j=3}^{i+1} m_j \right) + 1, \ldots, \sum_{j=3}^{i+2} m_j \right\}.$$

Each minor is a square submatrix of its respective level and each level is defined by differentiation of its respective variable. So, each of these minors is a Wronskian of monomials in one variable. Since each of these monomials in the top row of each level have different degree then all the monomials in the top row are linearly independent over the field $\mathbb{C}$. This is true only because there is no overlap. Since each of these minors is a Wronskian of monomial functions (i.e. analytic functions) of one variable and these functions are linearly independent we know that the determinant is a non-zero polynomial (see [13]). This monomial is the highest monomial in the determinant of $\mathcal{M}$ because the unique highest degree minor is chosen in each level. Note that this is also true only because there is no overlap. Thus, the determinant is a non-zero polynomial and the result is proved. □

In the next section we focus on characterizing when there is a unique highest degree monomial in the Laplacian decomposition of P even if there is overlapping of the first and second halves of each level.

II.5. Unique Laplace decomposition of $\texttt{in}(\det(\mathcal{M}))$

From now on we consider only the case where there is overlapping. We can find the highest degree monomial of the determinant similarly in this overlapping case, but the problem is that sometimes we can obtain the monomial in more than one way. Theoretically, these monomials can cancel each other out. We will completely characterize the case when the highest degree monomial is obtained in only one way from the Laplacian decomposition. First, notice that all of the columns outside the overlap are uniquely chosen as in the proposition. So, we only consider the columns in the overlap.

When $m_1 + m_2 \leq d$ there are columns of $\mathcal{M}$ that have the same degree entree in each row. By symmetry there are an even amount of these columns, say that number is $2r$. Label these columns from left to right $a_1, \ldots, a_r, b_1, \ldots, b_r$. If we restrict these columns to any level L_j then we have the following fundamental relationship of columns

$$p_j a_{i+1} = b_i$$

for all $1 \leq i < r$. So, over the polynomial ring in the variables $p_3, \ldots, p_n$ the columns a_{i+1} and b_i restricted to any level L_j are linearly dependent for all $1 \leq i < r$ and the minor containing those columns will be zero. The important rule that we use extensively in the remainder of the proof of the main theorem is that we can not choose columns a_{i+1} and b_i for any minor.

The following notation is extremely important in the remainder of the arguments. For more information about the determinant P we need to consider all columns of $\mathcal{M}$. Label all the columns of $\mathcal{M}$ numbered 1 through d. Then partition these columns $[d] = \{1, 2, \ldots, d-1, d\}$ into $n-2$ subsets (which will be the minors). Denote these partitions by $\overline{B}_3, \ldots, \overline{B}_n$, in increasing order each with m_i elements, respectively. Call the subset of overlapping columns O. Set $\hat{B}_i = O \cap \overline{B}_i$ and consider only those $\hat{B}_i$ which are non-empty. Put $\bar{m}_j = |\hat{B}_i|$. Then relabel these subsets of $[d]$ by $B_1, \ldots, B_k$ so that $B_1 = \hat{B}_h$ where h is the first index that gives $\hat{B}_i$ non-empty and $B_k = \hat{B}_u$ where u is the last index such that $\hat{B}_i$ is non-empty. We treat the collection $\{B_1, \ldots, B_k\}$ as a partition of the set of overlap columns $\{a_1, \ldots a_r, b_1, \ldots, b_r\}$. Make a partition with the B_i's by setting $J_1 = B_1$ and then for $q \geq 1$

$$J_{q+1} = B_{i_q+1} \cup B_{i_q+2} \cup \cdots \cup B_{i_{q+1}}$$

where for even q

$$i_q = min\{i \geq i_{q-1} + 1 : \chi_q \leq 2\}$$

and for odd q

$$i_q = min\{i \geq i_{q-1} + 1 : \chi_q \geq 0\}$$

where the characteristic function is given by

$$\chi_q = \sum_{i=1}^{q} (-1)^{i+1} |J_i|.$$

Within the proofs of the below facts we will choose the columns for each B_i so that we get the highest degree monomial in the determinant (i.e. we will be changing the members of each B_i, but this does not change the computations of the conditions). We need the following definitions to characterize when the highest degree monomial is uniquely represented in the Laplacian decomposition of the determinant.

Definition II.5.1. *A subpartition* $\{J_1, \ldots, J_q\}$ *is complete if* $\chi_q = 0$ *or* $\chi_q = 2$.

Definition II.5.2. *A subpartition* $\{J_1, \ldots, J_q\}$ *is diverse if* $\chi_q = 1$.

The goal here is to exhibit necessary and sufficient conditions for the uniqueness of the highest degree monomial in the Laplacian decomposition of the determinant. To accomplish this we create an algorithm with the following lemmas.

Lemma II.5.3. *The columns of a complete subpartition that do not contain a diverse subpartition or any other complete subpartition form the unique Laplacian decomposition for the highest degree non-zero monomial.*

Proof. Since there is no previous diverse or complete subpartitions $|J_{c+1}| > |\chi_c|$ for $c+1 < q$ where q is the last index in the complete subpartition as in Definition II.5.1. In particular, $|B_1| > 2$. To uniquely obtain the highest degree monomial from the minors we show that the elements of each J_i for $1 \leq i < q$ must be of the form

$$J_{i+1} = \{a_{k_i+1}, a_{k_i+2}, \ldots, a_{k_{i+1}}, b_{k_{i+1}}, b_{k_{i+1}+1}, \ldots, b_{\xi_{i+1}}\}.$$

This gives that

$$\bigcup_{j=1}^{i+1} J_j = \{a_1, \ldots, a_{k_{i+1}}, b_1, \ldots, b_{\xi_{i+1}}\}$$

where $\xi_i > k_i$ for all i (notice that for all $1 \leq i < q-1$ we have $k_{i+1} = \xi_i + 1$). We also prove that with the above hypothesis if N is even then $\chi_N = k_N - \xi_N$ and if N is odd then $\chi_N = \xi_N - k_N + 2$ for $N < q$. We do this by induction.

For the base case we know $J_1 = B_1$ and $|B_1| > 2$. So, $J_1 = \{a_1, b_1, \ldots, b_{\xi_1}\}$ since those are the highest degree columns which give the highest degree monomial in the Laplacian decomposition, it is unique, and $\xi_1 > k_1 = 1$. Then we can conclude $|J_1| = \xi_1 + 1 = \xi_1 - 1 + 2 = \xi_1 - k_1 + 2$.

For the second step we know that $|J_2| > |J_1|$ so the highest degree columns are $a_2, \ldots, a_{k_2}, b_{\xi_1+1}, \ldots, b_{\xi_2}$. We compute the characteristic:

$$\chi_2 = \xi_1 - k_1 + 2 - (k_2 - k_1 + \xi_2 - \xi_1) = 2\xi_1 - k_2 - \xi_2 + 2$$

$$= 2(k_2 - 1) - k_2 - \xi_2 + 2 = k_2 - \xi_2.$$

Also, $\chi_2 < 0$ which gives that $\xi_2 > k_2$. Now we assume the statement is true for all $j < 2N + 1$. In particular we have that $\chi_{2N} = k_{2N} - \xi_{2N}$ where $\xi_{2N} > k_{2N}$ and $\bigcup_{j=1}^{2N} J_j = \{a_1, \ldots, a_{k_{2N}}, b_1, \ldots, b_{\xi_{2N}}\}$. Then

$$J_{2N+1} = \{a_{k_{2N}+1}, \ldots, a_{k_{2N+1}}, b_{\xi_{2N}+1}, \ldots, b_{\xi_{2N+1}}\}$$

since these are the highest degree columns that give non-zero minors in the Laplacian decomposition and $2 < \chi_{2N+1} = \chi_{2N} + |J_{2N+1}|$. Then

$$\chi_{2N+1} = k_{2N} - \xi_{2N} + (k_{2N+1} - k_{2N} + \xi_{2N+1} - \xi_{2N}) = \xi_{2N+1} + k_{2N+1} - 2\xi_{2N}$$

$$= \xi_{2N+1} + k_{2N+1} - 2(k_{2N+1} - 1) = \xi_{2N+1} - k_{2N+1} + 2$$

and $2 < \chi_{2N+1} = \xi_{2N+1} - k_{2N+1} + 2$ which gives that $\xi_{2N+1} > k_{2N+1}$. Next we compute χ_{2N+2}. The only columns to get the highest degree monomial are $a_{k_{2N+1}+1}, \ldots, a_{k_{2N+2}}, b_{\xi_{2N+1}+1}, \ldots, b_{\xi_{2N+2}}$ since

$$0 > \chi_{2N+2} = \chi_{2N+1} - |J_{2N+2}|.$$

So, again we must choose some of the b_c columns. Then

$$\chi_{2N+2} = \xi_{2N+1} - k_{2N+1} + 2 - (k_{2N+2} - k_{2N+1} + \xi_{2N+2} - \xi_{2N+1})$$

$$= 2\xi_{2N+1} - k_{2N+2} + 2 - \xi_{2N+2}$$

$$= 2(k_{2N+2} - 1) - k_{2N+2} + 2 - \xi_{2N+2} = k_{2N+2} - \xi_{2N+2}$$

and again $0 > \chi_{2N+2} = k_{2N+2} - \xi_{2N+2}$ which gives $\xi_{2N+2} > k_{2N+2}$ as desired.

Now, we compute the columns for the last element in the complete subpartition, J_q. In either case of q being odd or even there are two cases of completeness: first when $|\chi_q| = 2$ or secondly $\chi_q = 0$. If $|\chi_q| = 2$ then $|J_q| = \xi_{q-1} - k_{q-1} + 2$. Thus, the only columns to choose to get the highest degree monomial are $a_{k_{q-1}+1}, \ldots, a_{k_q}, b_{\xi_{q-1}+1}$ and here notice that $k_q = \xi_{q-1} + 1$. If $\chi_q = 0$ then $|J_q| = \xi_{q-1} - k_{q-1}$. Thus, the only columns to choose to get the highest degree monomial are $a_{k_{q-1}+1}, \ldots, a_{\xi_{q-1}}$. □

The next Lemma will show when the combinatorics of the multiplicities permit more than one way to choose the highest degree monomial in the determinant.

Lemma II.5.4. *If a subpartition $\{J_1, \ldots, J_q\}$ is diverse and has no previous complete or diverse subpartitions and $|B_{i_q+1}| = 1$ or 2 then there is more than one non-zero choice to get the highest degree monomial in the Laplacian decomposition. Moreover, in this case there are exactly two non-zero choices to get the highest degree monomial in the Laplacian decomposition.*

Proof. We already have the setting for this lemma from Lemma II.5.3. Since there are no previous complete or diverse subpartitions by the claim in the above Lemma II.5.3 we know that $\bigcup_{j=1}^{q-1} J_j = \{a_1, \ldots, a_{k_{q-1}}, b_1, \ldots, b_{\xi_{q-1}}\}$ where $\xi_{q-1} > k_{q-1}$ and the columns for each J_j are chosen as in Lemma II.5.3. Since the subpartition is diverse $\chi_q = 1$. If q is even then $\chi_q = \chi_{q-1} - |J_q|$ and $\chi_{q-1} = \xi_{q-1} - k_{q-1} + 2$. So, we know $\xi_{q-1} - k_{q-1} + 2 - |J_q| = 1$ which shows that $|J_q| = \xi_{q-1} - k_{q-1} + 1$. The same happens if q is odd. In this case $\chi_q = \chi_{q-1} + |J_q|$ and $\chi_{q-1} = k_{q-1} - \xi_{q-1}$. So, we know that $k_{q-1} - \xi_{q-1} + |J_q| = 1$ which shows that $|J_q| = \xi_{q-1} - k_{q-1} + 1$.

First, we show that there is more than one way to get the highest degree monomial in the determinant when $|B_{i_q+1}| = 1$. We have already chosen the above columns for the minors before J_q. One choice is to take the columns $J_q = \{a_{k_{q-1}+1}, \ldots, a_{\xi_{q-1}+1}\}$ and then choose the column $b_{\xi_{q-1}+1}$ for the column of B_{i_q+1}. The other choice for columns that gives the same degree and is non-zero is $J_q = \{a_{k_{q-1}+1}, \ldots, a_{\xi_{q-1}}, b_{\xi_{q-1}+1}\}$ and take $a_{\xi_{q-1}+1}$ for the column of B_{i_q+1}. Thus, in this case there are two choices of minors in the Laplacian decomposition to get the highest degree monomial in the determinant.

Now assume that $|B_{i_q+1}| = 2$. One option is to take $J_q = \{a_{k_{q-1}+1}, \ldots, a_{\xi_{q-1}+1}\}$ and take $b_{\xi_{q-1}+1}$ and $b_{\xi_{q-1}+2}$ for the columns of B_{i_q+1}. The other option is to take $J_q = \{a_{k_{q-1}+1}, \ldots, a_{\xi_{q-1}}, b_{\xi_{q-1}+1}\}$ and take $a_{\xi_{q-1}+1}$ and $a_{\xi_{q-1}+2}$ for the columns of B_{i_q+1}. Hence, this case also provides two choices of minors in the Laplacian decomposition to obtain the highest degree monomial in the determinant. □

Lemma II.5.5. *If a subpartition* $\{J_1, \ldots, J_q\}$ *is diverse and has no previous complete or diverse subpartitions and* $|B_{i_q+1}| > 2$ *then there is only one choice of columns that realize the highest degree monomial in the determinant.*

Proof. We already have the set up from the previous two Lemmas II.5.3 and II.5.4. The choice to take $J_q = \{a_{k_{q-1}+1}, \ldots, a_{\xi_{q-1}}, b_{\xi_{q-1}+1}\}$ and to take $a_{\xi_{q-1}+1}$, $a_{\xi_{q-1}+2}$, $b_{\xi_{q-1}+2}$, $\ldots$, $b_{\xi_{q-1}+|B_i-q|-3}$ for the columns of B_{i_q+1} is a non-zero combination to get the highest degree monomial in the determinant. Now if we try the other combination where $J_q = \{a_{k_{q-1}+1}, \ldots, a_{\xi_{q-1}+1}\}$ and to take $b_{\xi_{q-1}+1}, \ldots, b_{\xi_{q-1}+|B_{i_q+1}|}$ for the columns of B_{i_q+1} then the degree of the monomial for B_{i_q+1} is smaller than the above so that we don't get the highest degree monomial. $\square$

Now we have the components to find exactly when the choice of columns is unique for each level in the matrix to give the highest degree monomial in the determinant. We do this by calculating subpartitions. By Lemma II.5.3 if a subpartition is complete then we can stop there, disregard those columns, and start over from there calculating subpartitions because they are uniquely chosen. Next, if we find a diverse subpartition after deleting the complete subpartitions there are two cases. First, by Lemma II.5.4 if the next multiplicity is 1 or 2 then the highest degree monomial can be attained in more than one way and we stop. Secondly, by Lemma II.5.5 if the next multiplicity is greater than two then we can disregard those columns and again start over. However, in this last case we choose the last column $b_{\xi_{q-1}+1}$ so after this there is no over lap on the column $a_{\xi_{q-1}+1}$ and we can recalculate the over lap and choose new B's, call them $\tilde{B}_i$, so that the first $\tilde{B}_1 = B_{i_q+1} \backslash \{a_{\xi_{q-1}+1}\}$ then the rest of the $\tilde{B}_i$'s are the same as before.

We have characterized the cases where the highest degree monomial of P is unique in the Laplacian decomposition. In these cases we have shown that this monomial has a non-zero coefficient. So, in these cases Theorem II.4.1 is proved. Thus, we may restrict to the case where there are more than one decompositions, sum them together, and show this sum is not zero. This is the subject of the next section.

II.6. Coefficient of the non-unique largest monomial

In this section we find the coefficient for $\texttt{in}(\det(\mathcal{M}))$ (in lexicographic ordering) and show that it is not zero. In this section the assumption that the multiplicities are all greater than 1 (i.e. $m_i \geq 2$ for all $1 \leq i \leq n$) is very important. Because of Lemmas II.5.3, II.5.4, and II.5.5 in the above section we just need to prove the case of Lemma II.5.4. In the proof we will need the following definitions.

Definition II.6.1. *Call a minor chosen in a level of the matrix $\mathcal{M}$ conservative if all the columns are consecutive.*

Definition II.6.2. *Call a minor chosen in a level of the matrix $\mathcal{M}$ liberal if the columns are not consecutive.*

Lemma II.6.3. *Assume the multiplicities, $[m_1, \ldots, m_n]$, satisfy that $m_i \geq 2$ for all $1 \leq i \leq n$ and there is a diverse subpartition with the next restricted multiplicity equal to one. Then $\texttt{in}(\det(\mathcal{M}))$ has a non-zero coefficient.*

Proof. We prove the result by cases. First, we consider the case where there is a diverse subpartition and the next restricted multiplicity is one. Since the multiplicities are ordered and they are all greater than one we know that this multiplicity must be the last multiplicity intersecting the overlap. So, with the notion from Section II.5 we have that $\chi_q = 1$ and $\bar{m}_u = 1$ where q is the second to last index. We have numbered the columns of $\mathcal{M}$ as $1, \ldots, d$. Assume the last overlap column b_r is column

h. Now, we show that there are exactly two ways to realize the largest monomial. By arguments in Section II.5 the last column chosen in the first half of $\mathcal{M}$ (which is given by f) is column number $h-r-m_{u-1}$. If this column is in the overlap then it is has column label $a_{r-m_{u-1}}$. The diversity of the partition gives two choices to realize the largest degree monomial in the determinant.

The first is to take the columns $[h-r-m_{u-1}+1, h-r-m_{u-1}+2, \ldots, h-r-1, h-r]$ for the minor in the $(u-1)^{th}$ level L_{u-1} and take the columns $[h, h+1, \ldots, h+m_u-1]$ for the u^{th} level L_u call this combo 1. The second choice is to take the columns $[h-r-m_{u-1}+1, h-r-m_{u-1}+2, \ldots, h-r-1, h]$ for the minor in the $(u-1)^{th}$ level L_{u-1} and take the columns $[h-r, h+1, \ldots, h+m_u-1]$ for the u^{th} level L_u call this combo 2.

Both of these minors have the same degree monomial since the columns h and $h-r$ have the top entree of the same degree. If any other columns are chosen the degree will drop and we will not realize the largest monomial in the determinant. Next, we compute the coefficients for each minor. Using Lemma II.4.2 the coefficient for the minor in the $(u-1)^{th}$ level in combo 1 is

$$\pm \prod_{m_1 \leq t < s \leq m_1+m_{u-1}-1} (s-t).$$

Also, using the Lemma II.4.2 the coefficient for the minor in the u^{th} level in combo 1 is

$$\pm \prod_{m_1-m_u \leq a < b \leq m_1-1} (b-a).$$

Again the coefficient for the minor in the $(u-1)^{th}$ level in combo 2 is

$$\pm \prod_{m_1+1\le c<d\le m_1+m_{u-1}-1}(d-c) \prod_{m_1+1\le z\le m_1-m_{u-1}-1}(z-(m_1-1)).$$

Finally, the coefficient for the minor in the u^{th} level in combo 2 is

$$\pm \prod_{m_1-m_u\le k<l\le m_1-2}(l-k) \prod_{m_1-m_u\le y\le m_1-2}(m_1-y).$$

The Laplacian decomposition gives that the coefficient for this monomial will be

$$\pm \prod_{m_1\le t<s\le m_1+m_{u-1}-1}(s-t) \prod_{m_1-m_u\le a<b\le m_1-1}(b-a)$$
$$\pm \left(\prod_{m_1+1\le c<d\le m_1+m_{u-1}-1}(d-c) \prod_{m_1+1\le x\le m_1-m_{u-1}-1}(z-(m_1-1)) \right.$$
$$\left. \prod_{m_1-m_u\le k<l\le m_1-2}(l-k) \prod_{m_1-m_u\le y\le m_1-2}(m_1-y) \right).$$

We want to show that this sum of products can not be zero. Factor out the following from each product

$$\prod_{m_1+1\le s<t\le m_1+m_{u-1}-1}(t-s) \prod_{m_1-m_u\le a<b\le m_1-2}(b-a).$$

After factoring this out we get the following sum

$$\pm \prod_{m_1+1\le s\le m_1+m_{u-1}-1}(s-m_1) \prod_{m_1-m_u\le t\le m_1-2}(m_1-1-t)$$
$$\pm \prod_{m_1+1\le x\le m_1-m_{u-1}-1}(z-(m_1-1)) \prod_{m_1-m_u\le y\le m_1-2}(m_1-y)$$

$$= \pm(m_{u-1}-1)!(m_u-1)! \pm m_{u-1}!m_u!.$$

Then we factor out the $(m_{u-1}-1)!(m_u-1)!$ from both terms and get

$$\pm 1 \pm m_{u-1}m_u$$

which is never zero since both m_{u-1} and m_u are greater than one. $\square$

Now, we consider the case where the next multiplicity after the diverse subpartition is two. The next lemma is the crux of the proof of the main theorem, and its proof is interminable.

Lemma II.6.4. *Assume the multiplicities $[m_1, \ldots, m_n]$ satisfy that $m_i \geq 2$ for all $1 \leq i \leq n$ and there is a diverse subpartition with the next restricted multiplicity equal to two. Then* $\texttt{in}(\det(\mathcal{M}))$ *has a non-zero coefficient.*

Proof. Again, since the multiplicities are ordered, the only multiplicities after the diverse subpartition are all two except the last overlapping multiplicity is $\bar{m}_u = 1$. Assume that the number of restricted multiplicities after the diverse subpartition is equal to $z+1$. The plan here is to find all the different ways to realize the largest monomial, then find the coefficients for each and show their sum is not zero. We will be choosing minors of size two inside each level after the diverse subpartition.

To realize the largest monomial the columns in the minors of size two after the diverse subpartition will either be conservative with all a columns, conservative with all b columns, or liberal with one a column and one b column. In the case that the 2×2 minor is conservative with all a columns it will be of the following form with determinant

$$\begin{vmatrix} p_j^t & p_j^{t-1} \\ tp_j^{t-1} & (t-1)p_j^{t-2} \end{vmatrix} = (-1)p_j^{2t-2}$$

for some t. In the case that the 2×2 minor is conservative with all b columns then it will be of the following form with determinant

$$\begin{vmatrix} p_j^t & p_j^{t-1} \\ (t-1)p_j^{t-1} & (t-2)p_j^{t-2} \end{vmatrix} = (-1)p_j^{2t-2}$$

for some t. So, in both of these cases the coefficient contributed to any realization of the largest monomial that uses this minor is -1. In the case where the 2×2 minor is liberal we have the following form with determinant

$$\begin{vmatrix} p_j^t & p_j^{t-1} \\ tp_j^{t-1} & (t-2)p_j^{t-2} \end{vmatrix} = (-2)p_j^{2t-2}$$

for some t. So, in this case the coefficient contributed to this particular realization of the largest monomial that uses this minor is -2. In the Laplace decomposition we assign a plus or minus to each of these minors according to where they are in the matrix. Since the columns and the rows of the conservative minors are consecutive the assigned term is plus one. However, for the liberal minors there may be a sign change which we will discuss later.

The next argument is very detailed. We describe and prove all the different possibilities for sequences of conservative and liberal minors that will realize the largest monomial given the diverse subpartition with the next restricted multiplicity equal to two. There will be many different sequences. We will construct a rooted tree that will show how the sequences are linked (we orient the tree downwards since the minors are chosen starting from the top and moving downwards). Each path of this tree denotes one Laplacian decomposition and to each vertex we attach the coefficient of the corresponding minor. Then the sum of these different Laplacian decompositions is equivalent to summing all the products of the coefficients of each vertex on each path. This process is described in detail below.

Notice the last restricted multiplicity in the diverse subpartition is equal to m_{u-z-1}. Since the subpartition is diverse we have two choices to make the highest degree monomial minor. This is the same argument as the above case. We have already computed the coefficient of the monomial of this minor in both cases. After factoring out a large product of integers from each coefficient we get that if the minor is conservative then the remaining term is ± 1 and if the minor is liberal then the remaining term is $\pm m_{u-z-1}$.

The next choice for the 2×2 minor is conservative with all b columns if the previous minor is conservative. Then the next 2×2 minor will have at least one a column to get the largest degree minor in that level. So, there will be two choices for this third minor from the last diverse minor. The choices are to take all a columns for a conservative minor or take the liberal minor with one a column and the next b column. If the minor for L_{u-z-1} is liberal then there are again two choices for the next 2×2 minor, it can be either conservative using the next consecutive a columns or it can be liberal with taking the next a column and then jumping to the next b column. This choice process is exactly what happens in general.

If the the minor chosen is liberal then there are two choices for the next minor (given that this minor was not the second to last level L_{u-1}, or the last level L_u in the overlap). If the minor is conservative with all a columns then the next minor must also be conservative but with all b columns (again, given that this minor was not the second to last level L_{u-1}, or the last level L_u in the overlap). If the minor is conservative with all b columns then the next minor can be either liberal or conservative as with the case of a liberal minor (again, given that this minor was not the second to last level L_{u-1}, or the last level L_u in the overlap).

The last minor in the overlap is determined by the choice of the previous minors because it has one column outside the overlap which is uniquely chosen. However, this last minor can change between liberal or conservative depending on how many liberal and conservative minors were chosen before it. There is exactly an even number of conservative minors in each subpartition because the amount of overlapping columns is even. So, if there is an even number of conservative minors chosen before the last minor in the overlap then this last minor must be liberal. If there is an odd number of conservative minors then this last minor must also be conservative. Thus, the last minor is completely determined by the previous minors chosen. Since the columns for this last minor are always the first two columns in the cofactor of the previously chosen minors the sign attributed will be plus and thus not change the sign. So, the signs in the bottom of the tree will always be negative. Here is a summary of the rules that a sequence of liberal and conservative minors follow:

1. The first minor can be either conservative or liberal.

2. After any liberal minor there can be either a conservative or a liberal minor.

3. The choice after a conservative minor at level L_λ must be conservative unless the minor in the $L_{\lambda-1}$ level was also conservative and at that level there was no choice for a liberal minor.

To visualize these rules we create the ranked tree as described above. We view each of these realizations of the largest monomial of the determinant as a path in this ranked tree. The ranking is given by the levels of the matrix so that the j^{th} level consists of minors in the $(h-z+j-1)^{th}$ level $L_{h-z+j-1}$. Each vertex is a monomial and we can attach the coefficient that that minor will contribute to the monomial. Then vertices are connected if they form a sequence in the Laplacian decomposition of the largest monomial. Again each path is a distinct realization of the largest monomial in the determinant where each vertex corresponds to the minor chosen for that level. This tree is exactly described by the successive method of choosing minors by the above rules. Figure II.1 is an example of the tree where $z = 5$. In the diagram a vertex C stands for a conservative minor and an L vertex stands for a liberal minor.

Denote this tree $\bar{T}_z$ for any z. Notice that this tree is exactly the classic tree to model pairs of rabbit population growth. Thus, the number of distinct Laplacian

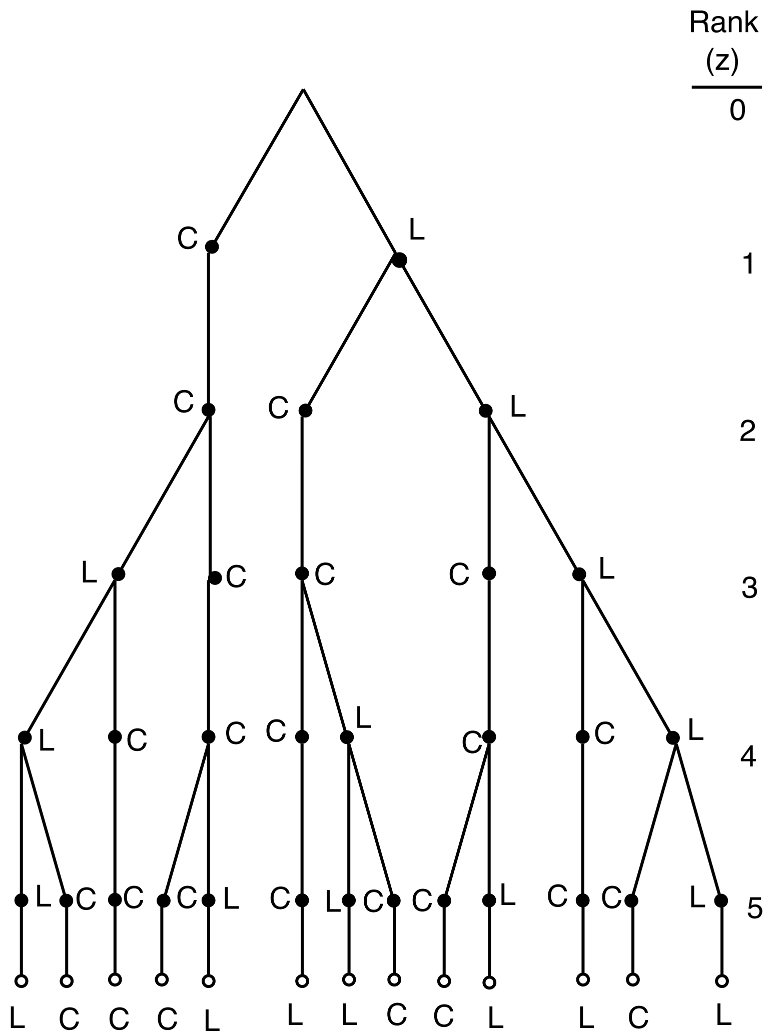

FIGURE II.1: Tree of the distinct realizations of `in`(det($\mathcal{M}$)) for $z = 5$.

decompositions of $\mathtt{in}(\det(\mathcal{M}))$ is given by the Fibonacci numbers. We can calculate the coefficient of the minor using this tree. Notice that we have not drawn the "full" tree. We have only drawn the part of the tree that corresponds to the non-unique realizations of the largest monomial (except that the top vertex corresponds to $z = 0$ where the realization is unique). So, we only calculate the factor of the coefficient that corresponds to the minors that are not uniquely chosen to realize the largest monomial in the determinant. To get the coefficient for each realization of the largest monomial of the determinant multiply all the coefficients for the minors on the corresponding path. Then the coefficient desired is the sum of the multiplication of all these paths. The aim here is to compute this sum. For any z call this sum $\bar{K}_z$. So, it only remains to show that $\bar{K}_z \neq 0$.

For calculation proposes define the tree T_z to be exactly $\bar{T}_z$ except that $m_{u-z-1} = 2$ and call this sum K_z. By Lemma II.4.2 we know that both $\bar{K}_z, K_z \in \mathbb{Z}$. The motivation for T_z is that $\bar{T}_z$ is built from T_z as in Figure II.2.

The structure of this tree produces the equation $\bar{K}_z = \pm m_{u-z-1} K_{z-1} \pm K_{z-2}$. Then with $m_{u-z-1} = 2$ we have that $K_z = \pm 2K_{z-1} \pm K_{z-2}$. We need to find the signs in these equations to show they are not zero. So, first examine the case where $m_{u-z-1} = 2$. The sign from the Vandermonde determinant formula from Lemma II.4.2 is factored out because it is the same for both choices of minors. So, the sign in the above equation only depends on the Laplacian decomposition and the sign of the minor.

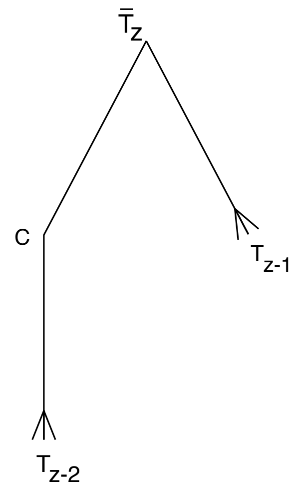

FIGURE II.2: The Tree $\bar{T}_z$ in terms of T_z.

In the Laplacian decomposition to find the sign contributed by a minor we sum the number of the columns and the number of the rows (not the amount, but the label numbers of the columns and rows). If this sum is odd then there is a negative sign attached and if the sum is even a positive sign is contributed (see [13]). This is why the conservative minors get sign negative one. Conservative minors have determinant negative one. Then the columns and rows are consecutive. So, with these conservative minors the sum of the numbers of the columns and rows is the sum of two even numbers and two odd numbers which is an even integer.

For the liberal minors the columns are not consecutive so there may be a negative sign attached. We sum the number of columns. Notice that this sum is taken in the relabeling of the rows and columns of the cofactor of all the previous taken minors. For each liberal minor, say we are at the $(u-z-1+j)^{th}$ level $L_{u-z-1+j}$, so that this is the j^{th} two after the diverse subpartition, the rows used are 1 and 2. The first column chosen is 1. Then the next column number is $z - j$ because there are $z - j$ many two size minors in the over lap after this minor. Thus, the signs of the liberal minors alternate from rank to rank in the tree. This gives us the following fundamental equation:

$$K_z = (-1)^z 2K_{z-1} + K_{z-2}.$$

We need an exact formula for K_z to show that $\bar{K_z} \neq 0$. First we find K_0 and K_1. K_0 is essentially calculated above except for the signs. This is the case where there is a diverse subpartition with the next restricted multiplicity equal to one, $m_{u-1} = 2$, and $m_u = 2$ (i.e. the case of non-uniqueness with no restricted multiplicities equal to two).

In Lemma II.6.3 we proved that after factoring out all similar terms the result is $\pm 1 \pm m_{u-1}m_u$. In this case the first sign is plus since it corresponds to the multiplication of two conservative minors which both have a negative sign. The second sign is negative because of the Laplacian decomposition. Both minors are liberal so they both have a negative sign but the first minor will get another negative sign because of the Laplacian decomposition. We skip a column where the second minor is forced to take the first column because it is at the end of the overlap. This arithmetic of the Laplacian decomposition is explained in the tree diagram which is Figure II.3. The extended branches of the end branches represent the uniquely chosen minors after the diverse subpartition. K_0 can be found by adding the products of the coefficients on each path. Thus, $K_0 = -3$.

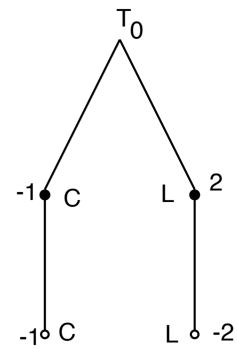

FIGURE II.3: The tree T_z for $z = 0$ with coefficients of each corresponding minor.

Similarly for $z = 1$ we can compute K_1 from the tree diagram in Figure II.4. We see that the signs on the liberal minors alternate. Thus, $K_1 = 5$.

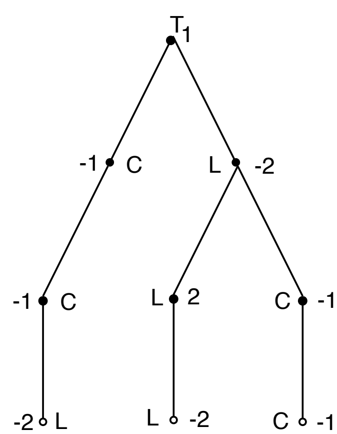

FIGURE II.4: The tree T_z for $z = 1$ with coefficients of each corresponding minor.

Then by an easy induction on z and using the above equation we get the following formula:

$$K_z = \begin{cases} 2z+3 & if\ z = 4w+1\ or\ z = 4w+2 \\ -2z-3 & if\ z = 4w+3\ or\ z = 4w. \end{cases}$$

Now we can compute $\bar{K_z}$. The same argument works for the signs of the minors here, so $\bar{K_z} = (-1)^z m_{u-z-1} K_{z-1} + K_{z-2}$. This gives that

$$\bar{K_z} = \begin{cases} -m_{u-z-1}(-2(z-1)-3) - 2(z-2) - 3 & if\ z = 4w+1 \\ m_{u-z-1}(2(z-1)+3) - 2(z-1) - 3 & if\ z = 4w+2 \\ -m_{u-z-1}(2(z-1)+3) + 2(z-1) + 3 & if\ z = 4w+3 \\ m_{u-z-1}(-2(z-1)-3) + 2(z-1) + 3 & if\ z = 4w \end{cases}$$

$$= \begin{cases} 2zm_{u-z-1} + m_{u-z-1} - 2z + 1 & if\ z = 4w+1 \\ 2zm_{u-z-1} + m_{u-z-1} - 2z + 1 & if\ z = 4w+2 \\ -2zm_{u-z-1} - m_{u-z-1} + 2z - 1 & if\ z = 4w+3 \\ -2zm_{u-z-1} - m_{u-z-1} + 2z - 1 & if\ z = 4w. \end{cases}$$

Since $m_{u-z-1} \geq 2$ all of these formulas are non-zero. □

Remark II.6.5. *The last line of the proof indicates the importance of the hypothesis that every multiplicity is greater than or equal to two. In the next section we drop this assumption and finish the proof of the main theorem.*

Remark II.6.6. *Notice that we have done more than just prove that the coefficient of* $\texttt{in}(\det(\mathcal{M}))$ *is not zero. We have actually computed this coefficient explicitly.*

Before moving to the next section let us remark about the case where $\sum_{i=1}^{n} m_i$ is odd. So far, under the main assumptions from the beginning of Section II.3 we have proved that both exponents are half the sum of the multiplicities. One of the assumptions was that the sum of the multiplicities was even. It turns out that it is fairly easy to prove the case when the sum of the multiplicities is odd. We do this by a deletion argument.

Suppose the sum $\tilde{n} = \sum_{i=1}^{n} m_i$ is odd and the points H_i are in general position. Also, assume that $\tilde{n} > 2n$. We can do this because we assumed that $m_i > 1$ for all i and in Section II.2 we prove a formula for the exponents for when $m_i = 2$ for all i. This assumption together with $m_i \geq 2$ for all i means there exists an $1 \leq j \leq n$ such that $m_j > 2$. Now, form a new multiarrangement by deleting a multiplicity from the point H_j. So, the new arrangement, call it $\tilde{\mathcal{A}}'$, has the same points as $\tilde{\mathcal{A}}$, but its multiplicity vector is $[m_1, \ldots, m_{j-1}, m_j - 1, m_{j+1}, \ldots, m_n]$. Then the sum of the multiplicities for $\tilde{\mathcal{A}}'$ is even and we can use the result from Lemma II.6.4 already proved about the exponents.

Suppose that there exists a derivation θ of minimal degree in $D(\tilde{\mathcal{A}})$. Then we know $\theta \in D(\tilde{A}')$ because these two multiarrangements are exactly the same except for the j^{th} multiplicity and $\theta(x - p_j y) \in (x - p_j y)^{m_j} S \subset (x - p_j y)^{m_j - 1} S$. Thus, if the points are in general position then Lemma II.6.4 shows that the degree of θ must be $(\tilde{n} - 1)/2$. Given the generalized Saito's criterion we have the exponents

$$\exp(\tilde{\mathcal{A}}) = (e_1, e_2) = ((\tilde{n} - 1)/2, (\tilde{n} + 1)/2).$$

II.7. Proof of the main theorem

We have proved all cases except if there are some multiplicities equal to one. So, in this section we drop the assumption that $m_i > 1$ for all i. The arguments of this section would not have been developed without the guidance of Sergey Yuzvinsky. In this section we change the assumptions slightly to allow for a completely general statement of the main theorem. These assumptions on the multiplicity vector, $[m_1, \ldots, m_n]$, are the following:

$$m_1 < \sum_{i=2}^{n} m_i \tag{II.1}$$

and

$$\sum_{i=1}^{n} m_i > 2n - 1. \tag{II.2}$$

Assumption II.1 was needed to construct the matrix and keep the exponents at half the sum of the multiplicities. Below we will finally see why Assumption II.2 is important. The focus is when there are multiplicities equal to one. Let α be the number of multiplicities equal to one. We finish the proof by induction on α. The arguments in Section II.6 prove the base case where $\alpha = 0$. Suppose that $\alpha > 0$. Since the multiplicities are ordered and $\alpha > 0$ we have $m_n = 1$. Again we use a deletion type argument so we form the multiarrangement defined by the same as $\tilde{\mathcal{A}}$ but we delete the last point. Call this new multiarrangement $\tilde{\mathcal{A}}' = \{H_1, \ldots, H_{n-1}\}$. Then $\tilde{\mathcal{A}}'$ has multiplicities $[m_1, \ldots, m_{n-1}]$. If $\sum_{i=1}^{n} m_i$ is odd then we can apply the deletion argument of Section II.6 since we have inductive hypothesis on α and the points are still in general position. Thus, we concentrate on the case where $\tilde{n} = \sum_{i=1}^{n} m_i$ is even. To proceed we make a definition.

Definition II.7.1. *A multiarrangement's exponents degenerate if* $\exp(\tilde{\mathcal{A}}) = (e_1, e_2)$ *where* $e_1 < \left\lfloor \frac{\tilde{n}}{2} \right\rfloor$.

Since we have assumed $\tilde{n} = \sum_{i=1}^{n} m_i$ is even we can again construct the matrix $\mathcal{M}$ (the only difference is that some of the levels will have height one). If the exponents degenerate then the determinant of $\mathcal{M}$ is zero (i.e. $P = 0$). This means that for every multiarrangement with multiplicities $[m_1, \ldots, m_n]$ the smallest exponent e_1 must satisfy

$$e_1 \leq \frac{\tilde{n}}{2} - 1.$$

Now, fix a multiarrangement $\tilde{\mathcal{A}}'$ with multiplicity vector $[m_1, \ldots, m_{n-1}]$ where the points are in general position. Then attaching another point, call it q, to $\tilde{\mathcal{A}}'$ with multiplicity one we get a multiarrangement with multiplicity vector $[m_1, \ldots, m_n]$. Call a new arrangement of this form $\tilde{\mathcal{A}}'_q$. Using the inductive hypothesis on $\tilde{\mathcal{A}}'$ we have that $\exp(\tilde{\mathcal{A}}') = \left(\frac{\tilde{n}}{2} - 1, \frac{\tilde{n}}{2}\right)$. Suppose that θ is the unique, up to a constant multiple, derivation in $D(\tilde{A}')$ that has degree $\frac{\tilde{n}}{2} - 1$. Since it is unique it must also be the unique derivation of that degree such that for any q we have $\theta \in D(\tilde{\mathcal{A}}'_q)$. So, if $q = \ker(\alpha_q)$ then $\theta(\alpha_q) \in (\alpha_q)S$ for any q. Thus, the derivation θ is in the derivation module for an arbitrary general position hyperplane arrangement (where all the multiplicities are one and the points are in general position). Call an arrangement of this form β. We can choose β so that $|\beta| > \frac{\tilde{n}}{2}$. Then $\exp(\beta) = (1, e)$ where $e > \frac{\tilde{n}}{2}$ and the 1 corresponds to the degree of the Euler derivation $\theta_E = x\partial_x + y\partial_y$ (see [14] Example 4.20).

We know that θ must be proportional to θ_E because $\deg(\theta) = \frac{\tilde{n}}{2} - 1 < \frac{\tilde{n}}{2}$. So, $\theta = T\theta_E$ for some homogeneous $T \in S$. From the arguments in Section II.2 the minimal (in degree) derivation in $D(\tilde{\mathcal{A}})$ proportional to θ_E is a constant multiple of the derivation $\bar{\theta} = \frac{\tilde{Q}}{Q}\theta_E$. Thus, the polynomial $\frac{\tilde{Q}}{Q}$ must divide T which implies that $\deg(T) \geq \deg(\bar{\theta}) = \tilde{n} - n$. We know by construction that $\deg(T) = \frac{\tilde{n}}{2} - 2$. Thus, we have the inequality

$$\frac{\tilde{n}}{2} - 2 \geq \tilde{n} - n$$

and this is equivalent to the inequality $\tilde{n} \leq 2n - 4$ which contradicts Assumption II.2. Thus, we have completed the proof: there can not be a derivation in $D(\tilde{\mathcal{A}})$ of degree $\frac{\tilde{n}}{2} - 1$ and the exponents must be $\left(\frac{\tilde{n}}{2}, \frac{\tilde{n}}{2}\right)$.

□

II.8. Degeneration varieties

As described above in Section II.7 given a multiplicity vector $[m_1, \ldots, m_2]$ that satisfies Assumptions II.1, II.2, and that $\sum_{i=1}^{n} m_i$ is even we can construct the matrix $\mathcal{M}$ and its determinant P. Recall, that P is a polynomial in the variables $p_3, \ldots, p_n$. More importantly, P has the property that if $P(\bar{z}) = 0$ for an $(n-2)$-tuple of complex numbers $\bar{z} = (z_3, \ldots, z_n)$ then the exponents of the multiarrangement defined by $\tilde{Q} = x^{m_1} y^{m_2} \prod_{i=3}^{n} (x - z_i)^{m_i}$ will degenerate in the terms of Definition II.7.1. This leads us to make the following definition.

Definition II.8.1. *The zero locus of a polynomial of the form of P is called a degeneration variety for the multiplicities $[m_1, \ldots, m_n]$.*

For the remainder of this section we study some examples of these degeneration varieties.

Example II.8.2. *This is the same as Example II.2.5 only now we compute the degeneration variety. The multiplicity vector is $[3, 3, 1, 1]$ so the defining polynomial is of the form $\tilde{Q} = x^3 y^3 (x - uy)(x - vy)$ where $u, v \in \mathbb{C} - 0$. Then the matrix $\mathcal{M}$ is*

$$\mathcal{M} = \begin{bmatrix} u^3 & u \\ v^3 & v \end{bmatrix}.$$

So, $P = uv(u-v)(u+v)$ and we re-scale coordinates so that $u = 1$ and the polynomial is now $P_{u=1} = v(1-v)(1+v)$. The parameter v can not take values 0 or 1 since that would change the multiplicity vector. However, $v = -1$ is a root of $P_{u=1}$ that does not change the multiplicity vector. Thus, we have justified the statements in Example II.2.5 that with $v = -1$ the exponents are $(3, 5)$, but if $v \neq -1$ the the exponents are $(4, 4)$.

Notice that in Example II.8.2 there were factors in P that would imply that the multiplicity vector would change. This is a general phenomenon since we can factor

p_i from each row of each level L_i and subtracting similar rows from different blocks and factoring out the difference $(p_i - p_j)$. Thus, we have shown that the determinant has the form

$$P = \left[\prod_{i=3}^{n} p_i^{m_i} \prod_{3 \le i < j \le n} (p_i - p_j)^{m_i}\right] \bar{P}.$$

Conjecture II.8.3. *The polynomial $\bar{P}$ is not divisible by any p_i or any $(p_i - p_j)$.*

The closest to a counterexample that the author has found to Conjecture II.8.3 is the following.

Example II.8.4. *In this example we study degeneration varieties for multiplicity vectors of the form $[3, 2, \ldots, 2, 1]$ where there are k many 2's. The calculations in this example were done on the computer program Macaulay 2 (see [9]). Each of the calculations satisfied Conjecture II.8.3, but the degeneration variety has interesting properties.*

- $[3, 2, 2, 1]$ *multiplicity vector gives* $\bar{P} = 2p_3 - p_4$.
- $[3, 2, 2, 2, 1]$ *multiplicity vector gives* $\bar{P} = 3p_3p_4 - (p_3p_5 + p_4p_5)$.
- $[3, 2, 2, 2, 2, 1]$ *multiplicity vector gives* $\bar{P} = 4p_3p_4p_5 - (p_3p_4p_6 + p_3p_5p_6 + p_4p_5p_6)$.
- $[3, 2, 2, 2, 2, 2, 1]$ *multiplicity vector gives* $\bar{P} = 5p_3p_4p_5p_6 - (p_3p_4p_5p_7 + p_3p_4p_6p_7 + p_3p_5p_6p_7 + p_4p_5p_6p_7)$.

The previous example led the author to make the following conjecture.

Conjecture II.8.5. *The degeneration polynomial for the multiplicity vector as in Example II.8.4 of the form* $[3, 2, \ldots, 2, 1]$ *where there are* k *many 2's is given by*

$$\bar{P} = kp_3p_4 \cdots p_{k+1} - \left(\sum_{j \neq k+2} p_3p_4 \cdots p_{j-1}p_{j+1} \cdots p_{k+2} \right).$$

II.9. Terao's conjecture

The lemmas in Section II.2 allow us to conclude Terao's conjecture for some particular cases of arrangements in $\mathbb{CP}^2$. The following is a list of the line arrangements in $\mathbb{CP}^2$ that have a restriction that satisfies the hypothesis of Lemma II.2.1, II.2.2, or II.2.3. These lemmas proved that the exponents of the restrictions are independent of the position of the points. Thus, Terao's conjecture holds (i.e. the freeness of the derivation module depends only on the intersection lattice) for all of the following arrangements:

1. There exists a line in $\mathcal{A}$ whose intersections with other lines of $\mathcal{A}$ are concentrated at no more than 3 points.

2. There exists a line in $\mathcal{A}$ such that all points on it have multiplicities not larger than 3.

3. There exists a line in $\mathcal{A}$ such that the multiplicity vector of the restricted multiarrangement satisfies $m_1 \geq \sum_{i=2}^{n} m_i$.

4. There exists a line in $\mathcal{A}$ such that the average of the multiplicities of the restricted multi-arrangement is less than 2.

If the freeness of these arrangements depends only on the intersection lattice then one can raise the following interesting question: for the above arrangements what are the characteristics of the intersection lattice that imply the derivation module is free? We extend this question to the following conjecture.

Conjecture II.9.1. *Suppose $\mathcal{A}$ is a line arrangement in $\mathbb{CP}^2$ that satisfies one of the 4 conditions above. If $\pi(\mathcal{A}, t)$ factors with integer coefficients then $\mathcal{A}$ is free.*

Next, we use the above conditions to deal with general arrangements of small sizes.

Remark II.9.2. *If $\mathcal{A}$ is a line arrangement in $\mathbb{CP}^2$ such that $|\mathcal{A}| \leq 8$ then either condition 1 or 4 holds. Hence, for $|\mathcal{A}| \leq 8$ Terao's conjecture holds.*

In fact we can prove that a larger class of arrangements satisfy Terao's conjecture. First, we need a little notation. Let $p \in \mathbb{CP}^2$ be a point that is an intersection of some lines from $\mathcal{A}$. Denote by $m(p)$ the number of lines in $\mathcal{A}$ that pass through p. Now, the next theorem extends Remark II.9.2.

Theorem II.9.3. *Let $\mathcal{A}$ be a line arrangement in $\mathbb{CP}^2$. If there exists a intersection point p such that $m(p) > \frac{1}{2}(|\mathcal{A}| - 3)$ then Terao's conjecture holds for $\mathcal{A}$ (i.e. for all arrangements with the intersection lattice isomorphic to that of $\mathcal{A}$).*

Proof. Suppose p satisfies the condition and denote by $\ell_1, \ell_2, \ldots, \ell_k$ all the lines of $\mathcal{A}$ passing through p where $k = \bar{m}(p)$. There are two alternatives.

(i) There are two lines $\ell,\ \ell' \in \mathcal{A}$ such that $p' = \ell \cap \ell' \notin \ell_i$ for every i. Then the points $\ell \cap \ell_1, \ldots, \ell \cap \ell_k, p'$ are pairwise distinct. this implies that the number k' of intersection points on ℓ satisfies $k' \geq k + 1 > \frac{1}{2}(|\mathcal{A}| - 1)$ whence the average of the multiplicities of the restriction of $\mathcal{A}$ to ℓ is less than 2. The result follows from item 4 above.

(ii) Every point of intersection lies on a line passing through p. This implies that the intersection lattice of $\mathcal{A}$ is supersolvable (see [14], pp. 30,31). Since every arrangement with a supersolvable lattice is free ([14], Theorem 4.58) the result follows.

□

In particular, Remark II.9.2 generalizes to the following.

Corollary II.9.4. *Terao's conjecture holds for line arrangements in* $\mathbb{CP}^2$ *of cardinality less than 11.*

Proof. Indeed every arrangement with at most 10 lines either satisfies condition 2 above or the condition of Theorem II.9.3. □

CHAPTER III

APOLAR ALGEBRAS

III.1. Introduction

In this chapter we study the apolar algebra of the defining polynomial of an arrangement of hyperplanes. There has been a large amount of interest in when apolar algebras are complete intersection algebras. It is an open problem to classify the polynomials whose apolar algebra is a complete intersection (see [10] pp. 261). The main references for this section are [5] and [10]. Denote by $S = \mathbb{C}[x_1, \ldots, x_\ell]$ and $\bar{S} = \mathbb{C}[\partial_{x_1}, \ldots, \partial_{x_\ell}]$ the polynomial algebras on the symbols $x_1, \ldots, x_\ell$ and $\partial_{x_1}, \ldots, \partial_{x_\ell}$ respectively. We call S the usual polynomial algebra in the variables $x_1, \ldots, x_\ell$. Then identify the partial derivative $\dfrac{\partial}{\partial_{x_i}}$ with the symbol ∂_{x_i}. Now we call $\bar{S}$ the polynomial ring of differential operators with constant coefficients. With this identification we have that S is an $\bar{S}$ graded module defined by differentiation. The grading on both S and $\bar{S}$ is the total polynomial degree. We denote by $S_d = \{p|\deg(p) = d\}$ and $\bar{S}_d = \{\theta|\deg(\theta) = d\}$. We define the action on monomials and extend linearly. If $m = x_1^{a_1}x_2^{a_2}\cdots x_\ell^{a_\ell} \in S_{d_1}$ (i.e. $\sum_{i=1}^{n} a_i = d_1$) and $\theta = \partial_{x_1}^{b_1}\partial_{x_2}^{b_2}\cdots\partial_{x_\ell}^{b_\ell} \in \bar{S}_{d_2}$ (i.e. $\sum_{i=1}^{n} b_i = d_1$) then the usual partial differentiation is given by

$$\theta(p) = \begin{cases} 0 & \text{if there exists an } i \text{ such that } b_i > a_i \\ \prod_{i=1}^{\ell} b_i! \binom{a_i}{b_i} x_i^{a_i - b_i} & \text{otherwise} \end{cases}$$

This action makes S a graded $\bar{S}$ module. Now we can define our main object of study: the annihilator of a polynomial. In this chapter we will deal with a polynomial p who is not necessarily a product of linear forms but an arbitrary polynomial. When p is a product of linear forms we will state it or denote it by Q. Now we can define our main object of study, the annihilator of a polynomial.

Definition III.1.1. *For $p \in S$ let $I(p) = \mathrm{Ann}_{\bar{S}}(p) = (0 :_{\bar{S}} p) = \{\theta \in \bar{S} | \theta(p) = 0\}$.*

Since all the elements of $\bar{S}$ have constant coefficients the action on S is commutative (i.e. for all θ_1 and θ_2 in $\bar{S}$ and for all $p \in S$ we have $\theta_1(\theta_2(p)) = \theta_2(\theta_1(p))$). This shows that $I(p)$ is an ideal in the polynomial ring $\bar{S}$. The quotient ring $A(p) = \bar{S}/I(p)$ is called the apolar algebra of the polynomial p. There is a similar ring defined by Macaulay called "an inverse system" (see [6] and [12]). Notice that if $\deg(\theta) > \deg(p)$ then $\theta(p) = 0$. Thus, for $i > \deg(p)$ we have $I(p) = \bar{S}_i$. This means that the apolar algebra is finite dimensional over $\mathbb{C}$ and hence Artinian and has Krull dimension zero. These algebras have even more intriguing properties. First we need a few more definitions.

Definition III.1.2. *Let A be an arbitrary finite dimensional commutative graded algebra over an arbitrary field k. Then we say A is Gorenstein if*

$$\mathrm{Hom}(A, k) \cong A$$

where this is an isomorphism of k algebras.

Gorenstein algebras can be defined for non-Artinian algebras and even arbitrary commutative rings, but we do not consider these rings in this thesis. Gorenstein algebras have been studied in many different areas of algebraic geometry and commutative algebra (see [2]). It turns out that these apolar algebras are all the zero dimensional Gorenstein rings. A good reference for these topics is Eisenbud's Commutative Algebra (see [6]).

Theorem III.1.3 (Macaulay, [12])**.** *An algebra A is a zero dimensional Gorenstein ring if and only if there exists a polynomial $p \in S$ such that $A \cong A(p)$.*

Using Theorem III.1.3 and Definition III.1.2 we get the following corollary.

Corollary III.1.4. *If $A = A(p)$ is a zero dimensional Gorenstein ring then its Hilbert series is symmetric and has socle degree* $\deg(p)$.

Since $A(p)$ is finite dimensional the smallest number of generators of $I(p)$ is ℓ. When $I(p)$ is generated by ℓ elements we say $A(p)$ is a complete intersection. For the remainder of this chapter we study polynomials whose apolar algebra is a complete intersection.

III.2. Complete intersection apolar algebras

In this section we study a few tools that will detect whether or not an apolar algebra is a complete intersection. The following lemma is a fundamental method used to show that an apolar algebra is a complete intersection. In [6] and [10] the author found a vague knowledge of this lemma and its proof in one direction of the implication. However, the author could not find any knowledge of the more interesting reverse direction of this implication.

Lemma III.2.1. *The apolar algebra $A(p)$ is a complete intersection with the ideal generated as $I(p) = (\theta_1, \theta_2, \ldots, \theta_\ell)$ if and only if the following three conditions are satisfied:*

1. $\theta_1, \ldots, \theta_\ell \in I(p)$,
2. $\{\theta_1, \ldots, \theta_\ell\}$ *is regular sequence in* $\bar{S}$, *and*
3. $$\sum_{i=1}^{\ell} [\deg(\theta_i) - 1] = \deg(p).$$

Proof. Assume that $I(p) = (\theta_1, \ldots, \theta_\ell)$. Since $\bar{S}$ is a polynomial ring (hence Cohen-Macaulay) we know that for any proper ideal $I < \bar{S}$ we have that $\text{depth}(I) = \text{codim}(I)$. We know $I(p)$ is proper because it can not contain any constants. If $\{\theta_1, \ldots, \theta_\ell\}$ is not a regular sequence then the $\text{depth}(I(p)) = \text{codim}(I(p)) < n$. However, this is impossible because Theorem III.1.3 says $A(p)$ is a zero dimensional ring.

It is left to prove that $\sum_{i=1}^{n}[\deg(\theta_i)-1]=\deg(p)$. We do this by a Hilbert series argument. For simplicity let $\deg(\theta_i)=d_i$ for all i and $d=\sum_{i=1}^{\ell}(d_i-1)$. It is well known (e.g. see [6] Exercise 21.17) that the Hilbert series $H(A(p),t)$ satisfies

$$H(A(p),t)=\frac{\prod_{i=1}^{\ell}(1-t^{d_i})}{(1-t)^n}=\prod_{i=1}^{\ell}\frac{1-t^{d_i}}{1-t}=\prod_{i=1}^{\ell}(1+t+t^2+\cdots+t^{d_i-1})$$

$$=1+c_1t+\cdots+c_1t^{d-1}+t^d.$$

By Corollary III.1.4 the socle degree is $\deg(p)$. Finally, by the Hilbert series above we have that the socle degree is also $d=\sum_{i=0}^{\ell}d_i-1$. Thus, $\deg(p)=d=\sum_{i=0}^{\ell}d_i-1$.

Now, we prove the more interesting converse. Assume that there are operators $\theta_1,\ldots,\theta_\ell\in\bar{S}$ that satisfy conditions 1, 2, and 3. Along with $\deg(\theta_i)=d_i$ denote the ideal $(\theta_1,\ldots,\theta_\ell)$ by J. We want to show that $J=I(p)$. Let $B=\bar{S}/J$ and $A=A(p)=\bar{S}/I(p)$. From the above Hilbert series argument

$$H(B,t)=1+\cdots+t^d$$

where $d=\sum_{i=1}^{\ell}[d_i-1]$. Then for A by Corollary III.1.4 we have that

$$H(A,t)=1+\cdots+t^{\deg(p)}.$$

This gives that $\deg(p)=d$ is the socle degree of both A and B. Since B is a complete intersection algebra it is Gorenstein and zero dimensional (see [6], Corollary

21.19). Thus, by Theorem III.1.3 there exists $p' \in S$ such that $B = A(p')$ (i.e. $J = I(p')$). We know that $J \subset I(p)$. Next we show by contradiction that $I(p) \subset J$.

Suppose $\eta \in \bar{S}$ is an operator of smallest degree, say $\deg(\eta) = \bar{d}$, such that $\eta \in I(p)$ and $\eta \notin J$. This shows the coefficient of the Hilbert series of B at degree $\bar{d}$, say $b_{\bar{d}}$, is strictly greater than the coefficient of the Hilbert series of A at degree $\bar{d}$, say $a_{\bar{d}}$ (i.e. $b_{\bar{d}} > a_{\bar{d}}$). Then by symmetry of the Hilbert series we know that $b_{d-\bar{d}} > a_{d-\bar{d}}$. Suppose that $\tilde{\eta} \in I(p)$ corresponds to this lowering of the Hilbert series coefficient at degree $d - \bar{d}$. So, $\tilde{\eta} \notin J$ and again by symmetry of the Hilbert series for all degrees $k > d - \bar{d}$ the ideals are equal (i.e. $I(p)_k = J_k$). Notice that $\bar{d} > 0$ otherwise the socle degree of A and B would not be the same. Thus, $\alpha\tilde{\eta} \in J_{d-\bar{d}+1}$ for all linear differential operators $\alpha \in \bar{S}_1$. We know that $J = I(p')$ so $[\alpha\tilde{\eta}](p') = 0$ for all linear differential operators with constant coefficients $\alpha \in \bar{S}_1$. But this means that for all $\alpha \in \bar{S}_1$ the polynomial $\tilde{\eta}(p')$ applied to α is zero, $\alpha(\tilde{\eta}(p')) = 0$. This implies that this polynomial is identically zero so $\tilde{\eta}(p') = 0$. This is a contradiction since we assumed that $\tilde{\eta} \notin J = I(p')$ and we have finished the proof.

□

Another tool used to detect if an arrangement's apolar algebra is a complete intersection is the catalecticant matrices. A good reference for catalecticant matrices, catalecticant determinants, and catalecticant varieties is [7] and [10].

Definition III.2.2. *Given a polynomial p of degree d and an integer k define the catalecticant operator*

$$ap_k : \bar{S}_k \longrightarrow S_{d-k}$$

by

$$ap_k(\theta) = \theta(p).$$

By construction $\ker(ap_k) = I(p)_k$. Also, ap_k is a linear map hence it is a

$$\binom{\ell + k - 1}{\ell - 1} \times \binom{\ell + d - k - 1}{\ell - 1}$$

size matrix. We denote by $\mathrm{Cat}_d(p, k)$ the matrix of ap_k and call it the k^{th} catalecticant of p. Note that $\mathrm{Cat}_d(p, k)$ is the zero matrix for $k > d$. If d is even, so $d = 2k$, then the matrix $\mathrm{Cat}_{2k}(p, k)$ is a square matrix. Since $\ker(\mathrm{Cat}_{2k}(p, k)) = I(p)_k$ we have that if there is no kernel or equivalently $\det(\mathrm{Cat}_{2k}(p, k)) \neq 0$ then $I(p)_k = 0$. If $d = 2k$ and $I(p)_k = 0$ then $A(p)$ can not be a complete intersection by Lemma III.2.1 (since if it was then the smallest degree generator would have degree greater than $\frac{d}{2}$, but there must be more than two generators ($\ell \geq 3$) so the sum of the degrees of these generators will be strictly greater than d). This reasoning is an advantageous method to show when an apolar algebra is not a complete intersection. We use this method in the next two sections. First, we study a few examples.

Example III.2.3. *Let $p = ax^2 + bxy + cxz + dy^2 + eyz + fz^2$ be any quadratic polynomial. So, we treat the symbols a, b, c, d, e as variables. By Lemma III.2.1 the only possible set of degrees of the generators of $I(p)$ is $(1, 2, 2)$ or $(1, 1, 3)$ if $A(p)$ is to* *be a complete intersection. If there is no degree 1 differential operator that annihilates p then p is not a complete intersection. Conversely, if there is a degree 1 differential operator that annihilates p then p is degenerate and we know that all apolar algebras in dimension two are complete intersections (see [10]). We have shown that p is a complete intersection if and only if* $Cat_2(p,1) = 0$. *So, we compute this determinant*

$$\mathrm{Cat}_2(p,1) = \begin{bmatrix} 2a & b & c \\ b & 2d & e \\ c & e & 2f \end{bmatrix}$$

and

$$\det(\mathrm{Cat}_2(p,1)) = 8adf - 2ae^2 - 2b^2f - 2c^2d + 2bce.$$

Notice that when $\det(\mathrm{Cat}_2(p,1)) = 0$ *is exactly when p is a degenerate quadratic form.*

Example III.2.4. *We know that all irreducible plane cubics up to projective isomorphism are either of the form* $x^3 + y^3 + z^3 - \lambda xyz$ *where* $\lambda \in \mathbb{C}$ *(there are equivalence classes given by the "j" invariant, see [3]), of the form* $x^3 + y^2z$*, or of the form* $x^3 + x^2 + y^2z$ *(again see [3]). It turns out that all of these isomorphism classes of irreducible plane cubics are complete intersections except for when* $\lambda = 0$ *and the cuspal cubic* $x^3 + y^2z$.

In the next section we begin to study the apolar algebra of an arrangement of hyperplanes.

III.3. Generic arrangement's apolar algebra

In this section we study generic arrangements. A generic arrangement is a central arrangement whose hyperplanes are in general position.

Example III.3.1. *Let* $p = xyz(ax+by+cz)$ *be an arrangement in* $\mathbb{C}^3$ *(so* $a, b, c \in \mathbb{C}$*). Then*

$$Cat_4(p,2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 2a & 0 & 2b & 2c \\ 0 & 2a & 0 & 2b & 2c & 0 \\ 0 & 0 & b & 0 & 0 & 0 \\ 2a & 2b & 2c & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The determinant is $\det(Cat_4(p,2)) = 8a^2b^2c^2$*. Thus,* $A(p)$ *is a complete intersection if and only if* a*,* b*, and* c *are all not zero. This shows that for generic arrangements in* $\mathbb{C}^3$ *of size four the apolar algebra is not a complete intersection.*

This example together with the fact that the apolar algebra of a generic polynomial is not a complete intersection (see [10]) suggests that the apolar algebra of a generic arrangement is not a complete intersection. Also, Example III.3.1 indicates that studying the catalecticant matrix of the defining polynomial of a generic arrangement may be a suitable method to show that its apolar algebra is not a complete intersection. However, the next example shows that the catalecticant matrices are complicated and may not present enough information for the proof.

Example III.3.2. *Let* $p = xyz(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z)(a_3x + b_3y + c_3z)$ *where we treat the coefficients* $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ *as variables over* $\mathbb{C}$*. For simplicity put*

$$p_1 = a_1a_2a_3,$$

$$p_2 = a_1a_2b_3 + a_1b_2a_3 + b_1a_2a_3,$$

$$p_3 = a_1a_2c_3 + a_1c_2a_3 + c_1a_2a_3.$$

$$p_4 = a_1b_2b_3 + b_1a_2b_3 + b_1b_2a_3,$$

$$p_5 = a_1b_2c_3 + a_1c_2b_3 + b_1a_2c_3 + b_1c_2a_3 + c_1a_2b_3 + c_1b_2a_3,$$

$$p_6 = a_1c_2c_3 + c_1a_2c_3 + c_1c_2a_3,$$

$$p_7 = b_1b_2b_3,$$

$$p_8 = b_1b_2c_3 + b_1c_2b_3 + c_1b_2b_3,$$

$$p_9 = b_1c_2c_3 + c_1b_2c_3 + c_1c_2b_3,$$

and

$$p_0 = c_1c_2c_3.$$

Then

$$Cat_6(p,3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 4p_1 & 0 & 0 & 2p_2 & 2p_3 & 0 \\ 0 & 0 & 12p_1 & 0 & 6p_2 & 6p_3 & 0 & 6p_4 & 4p_5 & 6p_6 \\ 0 & 12p_1 & 0 & 6p_2 & 6p_3 & 0 & 6p_4 & 4p_5 & 6p_6 & 0 \\ 0 & 0 & 6p_2 & 0 & 6p_4 & 4p_5 & 0 & 12p_7 & 6p_8 & 6p_9 \\ 24p_1 & 12p_2 & 12p_3 & 12p_4 & 8p_5 & 12p_6 & 24p_7 & 12p_8 & 12p_9 & 24p_0 \\ 0 & 12p_3 & 0 & 4p_5 & 6p_6 & 0 & 6p_8 & 6p_9 & 12p_0 & 0 \\ 0 & 0 & 2p_4 & 0 & 4p_7 & 2p_8 & 0 & 0 & 0 & 0 \\ 6p_2 & 6p_4 & 4p_5 & 12p_7 & 6p_8 & 6p_9 & 0 & 0 & 0 & 0 \\ 6p_3 & 4p_5 & 6p_6 & 6p_8 & 6p_9 & 12p_0 & 0 & 0 & 0 & 0 \\ 0 & 2p_6 & 0 & 2p_9 & 4p_0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The author used Macaulay 2 (see [9]) to compute the determinant of $Cat_6(p,3)$*. The result is a degree 30 polynomial in the variables* $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$*. This polynomial has approximately 1,800 terms so the author has decided to leave this polynomial out of the printing of these pages. However, the product of these variables* $a_1a_2a_3b_1b_2b_3c_1c_2c_3$ *does not factor out of the determinant. There are arrangements of this size that have complete intersection apolar algebras (see Theorem III.4.1) where some of the variables* a_i*,* b_i*, and* c_i *many be zero. We could always change coordinates so that any of these variables could be zero in these examples of complete intersection apolar algebras. Thus, the product not being a factor in the catalecticant determinant proves that the catalecticant determinant does not contain enough information to prove* *that the apolar algebra of a generic arrangement is not complete intersection.*

Even though Example III.3.2 is inconclusive the author still has faith in the following conjecture.

Conjecture III.3.3. *The apolar algebra of a generic arrangement is not a complete intersection.*

III.4. Complete intersection arrangements and free arrangements

In this section we discuss the relationship between arrangements of hyperplanes whose derivation module is free and arrangements whose apolar algebra is a complete intersection. We also study apolar algebras of reflection arrangements. To begin, we state a result from Kane's Reflection Groups (see [11]). Recall that a reflection arrangement is defined as the reflecting hyperplanes of the generating reflections.

Let G be a finite reflection group (i.e. a finite group generated by reflections in $GL_\ell(\mathbb{C})$). Denote $Q(G)$ to be the defining polynomial of the reflection arrangement for the group G with exponents given by the order of the corresponding reflection minus one (equivalently $Q(G)$ is the generator of the module of skew invariants of the group G, see [11]). Put I^G equal to the ideal in the polynomial ring S generated by the subring of invariants S^G. Similarly put $\bar{I}^G$ equal to the ideal in the polynomial ring of differential operators $\bar{S}$ generated by the subring of invariants $\bar{S}^G$. The next theorem can be deduced from arguments in [11].

Theorem III.4.1 (Kane, [11])**.** *The apolar algebra of $Q(G)$ is isomorphic to the algebra of coinvariants:*

$$\bar{S}/\bar{I}^G \cong \bar{S}/I(Q(G)).$$

In particular, $A(Q(G))$ is a complete intersection.

We show that there is hope that if $p = Q(G)q$ where q is a invariant of G then $A(p)$ will also be a complete intersection. We start by extending Theorem III.4.1.

Proposition III.4.2. *If $q \in I^G$ and $\theta \in \bar{I}^G_d$ where $d > \deg(q)$ then $\theta \in I(qQ(G))$.*

Proof. $\theta(qQ(G))$ is a skew symmetric polynomial because for all $g \in G$

$$g(\theta(qQ(G))) = [g(\theta)](g(qQ(G))) = \theta((\det g)^{-1}qQ(G))$$

$$= (\det g)^{-1}\theta(qQ(G))$$

since $Q(G)$ is skew symmetric, $q \in I^G$, and $\theta \in \bar{I}^G_d$. The module of all skew symmetric forms of G is generated by $Q(G)$ (see [11]). Since this polynomial is skew symmetric it must be a multiple of $Q(G)$ so $\theta(qQ(G)) = hQ(G)$ for some $h \in S$. However, by counting degrees $\deg(\theta(qQ(G))) = \deg(Q(G)) + \deg(q) - d < \deg(Q(G))$ since by hypothesis $d > \deg(q)$. Therefore, $h = 0$, $\theta(qQ(G)) = 0$, and $\theta \in I(qQ(G))$.

□

Now, we study examples of the type from Proposition III.4.2.

Example III.4.3. *For simplicity define $x := x_1$, $y := x_2$, and $z := x_3$. Let the polynomial $p = xyz(x-y)(x-z)(y-z)(x+y+z)$ be the defining polynomial for the* *arrangement $\mathcal{A}$. The following is a picture of this arrangement as a line arrangement in $\mathbb{RP}^2$ where the circle is the hyperplane at infinity given by $z = 0$.*

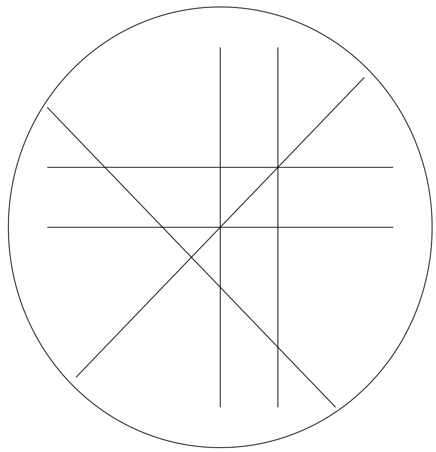

FIGURE III.1: The braid arrangement with an additional line $x + y + z$

It turns out that $I(p) = (\theta_1, \theta_2, \theta_3)$ where

$$\theta_1 = \partial_x^2\partial_y + \partial_x\partial_y^2 + \partial_x^2\partial_z + \partial_x\partial_y\partial_z + \partial_y^2\partial_z + \partial_y\partial_z^2,$$

$$\theta_2 = \partial_x^3 + \partial_y^3 + \partial_z^3,$$

$$\theta_3 = \partial_x^4 + \partial_x^3\partial_y + \partial_x^2\partial_y^2 + \partial_x^3\partial_z + \partial_x^2\partial_z^2 + \partial_y^3\partial_x + \partial_y^4 + \partial_y^3\partial_z + \partial_y^2\partial_z^2 + \partial_y\partial_z^3 + \partial_x\partial_z^3 + \partial_z^4.$$

The operators θ_1, θ_2, and θ_3 are in $I(p)$ by Proposition III.4.2. We show they form a regular sequence by a Macaulay 2 calculation:

```
Macaulay 2, version

--Copyright 1993-2001, D. R. Grayson and M. E. Stillman

--Singular-Factory 1.3b, copyright 1993-2001, G.-M. Greuel, et al.

--Singular-Libfac 0.3.2, copyright 1996-2001, M. Messollen

i1 : S=QQ[dx,dy,dz]

o1 = S

o1 : PolynomialRing

i2 : I=ideal(dx^2*dy+dx^2*dz+dy^2*dx+dy^2*dz+dz^2*dx+dz^2*dy+dx*dy*dz,

dx^3+dy^3+dz^3,dx^4+dx^3*dy+dx^3*dz+dx^2*dy^2+dx^2*dz^2+dy^4+

dy^3*dx+dy^3*dz+dy^2*dz^2+dz^4+dz^3*dx+dz^3*dy)

o2 : Ideal of S

i3 : A=S/I

o3 = A

o3 : QuotientRing

i4 : dim A

o4 = 0.
```

Since the dimension of the quotient ring is zero and there are three generators of the ideal then the generators must form a regular sequence. Then we just count the degrees $(3,3,4)$ *of the generators by* $2+2+3=7$ *which is the degree of* p*. Hence, by Lemma III.2.1 we know* $A(p)$ *is a complete intersection.*

Now, notice that the Poincaré polynomial of this arrangement is

$$\pi(\mathcal{A}, t) = 1 + 7t + 16t^2 + 10t^3 = (1 + t)(10t^2 + 6T + 1)$$

which does not completely factor. By Theorem I.3.8 we know that if the arrangements Poincaré polynomial does not factor then the arrangement can not be free. Thus, we have an arrangement of hyperplanes whose apolar algebra is a complete intersection but it is not free.

Remark III.4.4. *Notice that the polynomial p is skew symmetric for the reflection group of type A. Moreover, this arrangement is of the form $qQ(G)$.*

Now we look at a few examples similar to Example III.4.3, but they are not arrangements. However, it is still of interest to know some polynomials that admit complete intersection apolar algebras and they are also of the form $qQ(G)$. (The author does not know if these examples are free).

Example III.4.5. *In this example we study two polynomials that are similar to that of Example III.4.3, but they are not arrangements. Using the same arguments from Example III.4.3 we list the generators of the annihilator ideal. These calculations were assisted by the computer program Macaulay 2 (see [9]).*

- *Let $p = xyz(x-y)(x-z)(y-z)(x^2+y^2+z^2)$ then $I(p)$ is generated by*

  $$\theta_1 = \partial_x^3 + \partial_y^3 + \partial_z^3 + \partial_x^2\partial_y + \partial_x\partial_y^2 + \partial_x^2\partial_z + \partial_x\partial_z^2 + \partial_x\partial_y\partial_z + \partial_y^2\partial_z + \partial_y\partial_z^2,$$

  $$\theta_2 = \partial_x^2\partial_y^2 + \partial_x^2\partial_z^2 + \partial_y^2\partial_z^2,$$

  $$\theta_3 = \partial_x^4 + \partial_x^3\partial_y + \partial_x\partial_y^3 + \partial_x^3\partial_z + \partial_x\partial_z^3 + \partial_y^4 + \partial_z^4 + \partial_x^2\partial_y\partial_z + \partial_x\partial_y^2\partial_z + \partial_x\partial_y\partial_z^2.$$

- *Let* $p = xyz(x^2 - y^2)(x^2 - z^2)(y^2 - z^2)(x^2 + y^2 + z^2)$ *then* $I(p)$ *is generated by*

$$\theta_1 = \partial_x^2\partial_y^2 + \partial_x^2\partial_z^2 + \partial_y^2\partial_z^2,$$

$$\theta_2 = \partial_x^4 + \partial_y^4 + \partial_z^4,$$

$$\theta_3 = \partial_x^2\partial_y^4 + \partial_x^2\partial_z^4 + \partial_y^2\partial_x^4 + \partial_y^2\partial_z^4 + \partial_z^2\partial_x^4 + \partial_z^2\partial_y^4.$$

We conclude this section with a conjecture that is the generalization of these examples and the extension of Proposition III.4.2.

Conjecture III.4.6. *If* G *is a finite reflection group and* $q \in I^G$ *then* $A(qQ(G))$ *is a complete intersection.*

In the next section we study the combinatoriality of the property that the apolar algebra is a complete intersection.

III.5. Non-combinatoriality of the complete intersection property

The next example shows that the complete intersection property of an arrangement of hyperplanes is not a combinatorial property. Example III.4.3 and Example III.5.1 are two arrangements that have isomorphic intersection lattices but Example III.4.3 has a complete intersection apolar algebra and the apolar algebra of Example III.5.1 is not a complete intersection.

Example III.5.1. *In this example we just shift the special line* $x + y + z$ *in Example III.4.3 to* $x + y + 2z$. *So, the arrangement we study in this example is defined by* $p = xyz(x - y)(x - z)(y - z)(x + y + 2z)$.

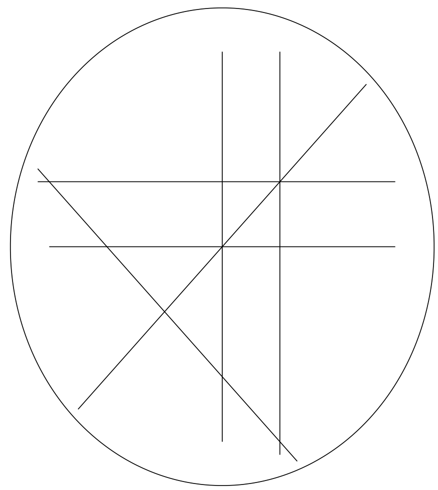

FIGURE III.2: The braid arrangement with the additional shifted line $x + y + 2z$

We use Catalecticant matrices to show that $A(p)$ *can not be a complete intersection. The kernel of the matrix* $Cat_7(p, 3)$ *represents all the possible differential* *operators in $I(p)$ of degree 3. The matrix is*

$$Cat_7(p,3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & -12 & 0 & 8 & -8 & 0 & 0 & 4 & -6 & 0 \\ 0 & -12 & 0 & 8 & -8 & 0 & 0 & 4 & -6 & 0 \\ 0 & 0 & 12 & 0 & 0 & 6 & 0 & -12 & -6 & 0 \\ 0 & 0 & 0 & 6 & -9 & 0 & -6 & 0 & 24 & 0 \\ 0 & 24 & -24 & 0 & 12 & -18 & -24 & -12 & 0 & 48 \\ 24 & 0 & 12 & -24 & -12 & 48 & 0 & 24 & 18 & -48 \\ -24 & 12 & -18 & -12 & 0 & 0 & 24 & 18 & -48 & 0 \\ -24 & 12 & -18 & -12 & 0 & 0 & 24 & 18 & -48 & 0 \\ 0 & 0 & 0 & 0 & -8 & -4 & 0 & 0 & 8 & 6 \\ 0 & -6 & 0 & 0 & 16 & 0 & 6 & -16 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & -8 & -4 & 0 & 8 & 6 & 0 & 0 & 0 & 0 \\ -6 & -6 & 0 & 12 & 9 & -24 & 0 & 0 & 0 & 0 \\ -6 & 0 & 16 & 6 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We compute the kernel of this matrix using the computer program Macaulay 2 (see [9]) and find that the kernel is zero. Thus, there are no degree 3 differential operators in $I(p)$. This shows that the smallest degrees of generators for $I(p)$ would be $(4,4,4)$.

Then we observe that the sum $(4-1)+(4-1)+(4-1) = 9$ *is strictly greater than seven. Thus, by Lemma III.2.1 we know that* $A(p)$ *can not be a complete intersection.*

This example proves that Terao's conjecture in the setting of complete intersection apolar algebras is false.

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