Qualifying Examination

# Theory of Probability

Winter 2006

**NAME:**______________________________

## Instructions:

(1) This is a **close** book and **close notes** exam.

(2) This examination consists a total of **SIX** questions and comprises **three** printed pages (including the cover page).

(3) Answer **all** questions in **3 hours**. Solve problems step by step and show all your work.

**Throughout this paper, all random variables are real-valued unless otherwise specified.**

1. (10 points) Let $\{X_t, t \geq 0\}$ be a standard one-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{\mathcal{F}_t \triangleq \sigma(X_s, s \leq t), t \geq 0\}$ be the natural $\sigma$-algebra flow generated by the Brownian motion $X$. Prove that :
   (1) $\{X_t, \mathcal{F}_t, t \geq 0\}$ is a martingale;
   (2) $\{X_t^2 - t, \mathcal{F}_t, t \geq 0\}$ is a martingale;
   (3) For any $a \in \mathbb{R}$, $\{e^{aX_t - \frac{a^2 t}{2}}, \mathcal{F}_t, t \geq 0\}$ is a martingale.

2. (15 points) Let $\{X_n, \mathcal{F}_n, n \geq 0\}$ be a supermartingale on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that for any $\lambda > 0$ and $n \geq 0$, we have

$$\lambda \mathbb{P}(\inf_{k \leq n} X_k < -\lambda) \leq \int_{\{\inf_{k \leq n} X_k < -\lambda\}} (-X_n) d\mathbb{P}.$$

3. (15 points) Prove that for any nonnegative, real-valued random variable $X$,

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > t) dt = \int_0^\infty \mathbb{P}(X \geq t) dt.$$

4. (20 points) Let $\{B_t, t \geq 0\}$ be a standard one-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $a > 0$ be any real number. Define $T_a = \inf\{t > 0 : B(t) = a\}$. Prove that
   (1)
$$\mathbb{P}(\max_{0 \leq s \leq t} B(s) \geq a) = \mathbb{P}(T_a \leq t) = \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^\infty e^{-y^2/2} dy;$$
   (2)
$$\mathbb{P}(T_a < \infty) = 1;$$
   (3)
$$\mathbb{E}(T_a) = \infty.$$
   (Hint: Using the result of question 3.)

5. (20 points) A biological problem can be described as follows. Each cell of a certain organism contains $N$ particles, some of which are of type $A$, the others of type $B$. The cell is said to be in state $E_j$, $j \leq N$ if it contains exactly $j$ particles of type $A$. Daughter cells are formed by cell division, but prior to the division each particle replicates itself; the daughter cell inherits $N$ particles chosen at random from the $2j$ particles of type $A$ and $2N - 2j$ particles of type $B$ present in the parental cell.
   (1) Argue that the state transition of daughter cells from generation to generation is a

Markov chain.

(2) Find the one step transition probability.

(3) Prove that this Morkov chain is also a martingale.

6. (20 points) A stochastic process $\{N(t), t \geq 0\}$ is said to be a counting process if $N(t)$ represents the total number of "events" that have occurred up to time $t$. Let $N(t)$ be a counting process satisfying following properties:

(1) $N(0) = 0$;

(2) For any $s > 0$ and $0 = t_0 < t_1 < t_2 < \cdots < t_n$, $\{(N(t_i) - N(t_{i-1})), i = 1, \cdots n\}$ are independent and for each $i$, $(N(t_i + s) - N(t_{i-1} + s))$ and $(N(t_i) - N(t_{i-1}))$ have same distribution;

(3) $\mathbb{P}(N(h) = 1) = \lambda h + \circ(h)$ with $\lambda > 0$, where $\circ(h)$ is a higher order infinitesimal of $h$

(4) $\mathbb{P}(N(h) \geq 2) = \circ(h)$.

Prove that for any $t, s \geq 0$,

$$\mathbb{P}(N(t+s) - N(s) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \qquad n \geq 0.$$

Thus, $\{N(t), t \geq 0\}$ is a Poisson process.