Probability Qualifying Examination

Summer 2009

University of Oregon

Department of Mathematics

STOP! Choose 8 problems from the 10 below.

Problem	Possible Points	Earned Points
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	80	

Problem 1. For X, Y two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, define $\rho(X, Y) := \mathbb{E}(\min\{|X - Y|, 1\})$. Prove that ρ is a metric on $(\Omega, \mathcal{F})$ such that $\rho(X_n, X) \to 0$ if and only of $X_n \xrightarrow{\text{Pr}} X$.

Solution to Problem 1. Suppose that $X_n \xrightarrow{\text{Pr}} X$. Then by definition, for all $\varepsilon > 0$,

$$\mathbb{P}(|X_n - X| > \varepsilon) \to 0\,.$$

Since $\min\{|X_n - X|, 1\} \leq 1$ on the event $\{|X_n - X| > \varepsilon\}$ and $\min\{|X_n - X|, 1\} \leq \varepsilon$ on the event $\{|X_n - X| \leq \varepsilon\}$, we have

$$\mathbb{E}(\min\{|X_n - X|, 1\}) \leq \mathbb{P}(|X_n - X| > \varepsilon) + \varepsilon\,.$$

Thus,

$$\lim_{n\to\infty} \mathbb{E}(\min\{|X_n - X|, 1\}) \leq \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) + \varepsilon = \varepsilon\,.$$

Since $\varepsilon > 0$ is arbitary, we have that

$$\lim_{n\to\infty} \mathbb{E}(\min\{|X_n - X|, 1\}) = 0\,. \tag{1}$$

Now suppose that (1) holds. We have that

$$\mathbb{E}(\min\{|X_n - X|, 1\}) = \mathbb{P}(|X_n - X| > 1) + \int_{|X_n - X| \leq 1} |X_n - X| d\mathbb{P} \geq \mathbb{P}(|X_n - X| > 1)\,,$$

whence $\lim_{n\to\infty} \mathbb{P}(|X_n - X| > 1) = 0$. Also,

$$\begin{aligned}\mathbb{P}(1 \geq |X_n - X| > \varepsilon) &= \mathbb{P}(|X_n - X| > \varepsilon \text{ and } |X_n - X| \leq 1) \\ &\leq \mathbb{P}(|X_n - X| \wedge 1 > \varepsilon) \leq \frac{\mathbb{E}(|X_n - X| \wedge 1)}{\varepsilon}\,.\end{aligned}$$

Thus, $\lim_{n\to\infty} \mathbb{P}(1 \geq |X_n - X| > \varepsilon) = 0$. ■

Problem 2. Let $\{X_k\}$ be an i.i.d. sequence of random variables with $\mathbb{E}X_i = 0$ and $\mathrm{Var}(X_i) = \sigma^2$. Let T be a stopping time for the filtration $\{\mathcal{F}_k\}$, where $\mathcal{F}_k = \sigma(X_1, \ldots, X_k)$. Let $S_n = \sum_{i=1}^n X_i$. Prove that if $\mathbb{E}T < \infty$, then

$$\mathrm{Var}(S_T) = \sigma^2 \mathbb{E}T\,.$$

Solution. Note that since $\mathbb{E}X_i = 0$, we have $\sigma^2 = \mathbb{E}X_1^2$. The process $n \mapsto S_n^2 - \sigma^2 n$ is a martingale, whence (since $\mathbb{E}T < \infty$),

$$\begin{aligned} 0 &= \mathbb{E}[S_{n\wedge T}^2 - \sigma^2(n \wedge T)] \\ &= \mathbb{E}S_{n\wedge T}^2 - \sigma^2\mathbb{E}(n \wedge T)\,. \end{aligned}$$

That is,

$$\mathbb{E}S_{n\wedge T}^2 = \sigma^2\mathbb{E}(n \wedge T)\,. \tag{2}$$

By the Monotone Convergence Theorem, the right-hand side of (2) converges to $\sigma^2\mathbb{E}T$. Therefore, the left-hand side, $\mathbb{E}S_{n\wedge T}^2$ is bounded. Since $n \mapsto S_n$ is a martingale, and so is $n \mapsto S_{n\wedge T}$, we conclude that $S_{n\wedge T}$ converges in L^2 and a.s., and the limit must be S_T, since $T < \infty$ a.s. Therefore,

$$\lim_{n\to\infty} \mathbb{E}S_{n\wedge T}^2 = S_T^2,$$

and taking the limit as $n \to \infty$ on both sides of (2) shows that

$$\mathbb{E}S_T^2 = \mathbb{E}X_1^2\mathbb{E}T\,. \tag{3}$$

■

Problem 3. Let $\{X_n\}$ be a martingale with $X_0 = 0$ such that the increment $\mathbb{E}(X_n - X_{n-1})^2 = c_1$ for some constant $c_1 > 0$. Show that $X_n/n \to 0$ almost surely.
Hint: First show that $X_{a_n}/a_n \to 0$ a.s. along a suitable subsequence $\{a_n\}$.

Solution to Problem 3. Since the increments of martingales are uncorrelated,

$$\mathrm{Var}(X_n) = \sum_{k=1}^{n} \mathrm{Var}(X_{n+1} - X_n) \leq c_1 n\,.$$

Since $\mathbb{E}X_n = 0$, by Chebyshev's inequality,

$$\mathbb{P}(|X_n| \geq n\varepsilon) \leq \frac{\mathrm{Var}(X_n)}{n^2\varepsilon^2} \leq \frac{c_1}{n\varepsilon^2}\,.$$

Note that

$$\mathbb{P}(X_{n^2} \geq n^2\varepsilon) \leq \frac{c_1}{n^2\varepsilon^2},$$

whence by the Borel-Cantelli Lemma, $|X_{n^2}/n^2| \leq \varepsilon$ eventually. Since $\varepsilon > 0$ is arbitrary, $X_{n^2}/n^2 \to 0$ as $n \to \infty$.

By Doob's inequality,

$$\mathbb{P}\left(\max_{n^2 \leq k \leq (n+1)^2} |X_k - X_{n^2}| > \varepsilon n^2\right) \leq \frac{c_2\mathbb{E}(X_k - X_{n^2})^2}{\varepsilon^2 n^4} \leq \frac{c_3(2n+1)}{\varepsilon^2 n^4} \leq \frac{c_4}{n^3}\,.$$

Thus, a.s., eventually

$$\max_{n^2 \leq k \leq (n+1)} \left|\frac{X_k}{n^2} - \frac{X_{n^2}}{n^2}\right| \leq \varepsilon\,.$$

Thus, a.s., for $n^2 \leq k \leq (n+1)^2$,

$$\left|\frac{X_k}{k}\right| \leq \left|\frac{X_k}{n^2}\right| \leq \left|\frac{X_k}{n^2} - \frac{X_{n^2}}{n^2}\right| + \frac{X_{n^2}}{n^2} \leq \varepsilon + \varepsilon$$

if n is large enough. ∎

Problem 4. Suppose that $\{X_j\}$ are i.i.d. with $\mathbb{E}X_j = 0$ and $\mathbb{E}X_j^2 = 1$. Prove that for $0 < r < 1$,

$$\sqrt{1-r}\sum_{j=0}^{\infty} r^j X_j \xrightarrow{\text{w}} N(0,\gamma^2)$$

as $r \to 1$, and find γ.

Hint: First show that $\sum_{j=0}^{\infty} r^j X_j$ converges a.s.

Solution to Problem 4. First note that

$$\mathrm{Var}\left(\sum_{j=0}^{n} r^j X_j\right) = \sum_{j=0}^{n} r^{2j},$$

and the latter sum converges, whence $\sum_{j=0}^{n} r^j X_j$ converges to a random variable $S^{(r)}$ a.s. Write $\varphi_n^{(r)}$ for the ch.f. of $\sqrt{1-r}\sum_{j=0}^{n} r^j X_j$, and $\varphi^{(r)}$ for the ch.f. of $\sqrt{1-r}\sum_{j=0}^{\infty} r^j X_j$. We have that $\varphi_n^{(r)}(t) \to \varphi^{(r)}(t)$ as $n \to \infty$, since the corresponding random variables converge a.s. (and hence in distribution).

We have that

$$e^{ix} = 1 + ix - \frac{x^2}{2} + \varepsilon(x),$$

where $|\varepsilon(x)| \le c[x^2 \wedge |x|^3]$.

$$\varphi_n^{(r)}(t) = \prod_{j=0}^{n} \mathbb{E}e^{ir^j\sqrt{1-r}X_j} = \prod_{j=0}^{n}\left[1 - \frac{(1-r)r^{2j}t^2}{2} + \varepsilon(r^j\sqrt{1-r})\right].$$

Since $|\prod_{j=0}^{n} z_j - \prod_{j=0}^{n} w_j| \le \sum_{j=0}^{n} |z_j - w_j|$ for complex numbers z_j, w_j in the unit disc, we have

$$\begin{aligned}
\left|\varphi_n^{(r)}(t) - \prod_{j=0}^{n}\left[1 - \frac{(1-r)r^{2j}t^2}{2}\right]\right| &\le \sum_{j=0}^{n} \varepsilon\left(r^j\sqrt{1-r}\right)\\
&\le c\sum_{j=0}^{n} \mathbb{E}\left[r^{2j}(1-r)X^2 \wedge r^{3j}(1-r)^{3/2}|X|^3\right]\\
&\le c(1-r)\sum_{j=0}^{n} r^{2j}\mathbb{E}[X^2 \wedge \sqrt{1-r}|X|^3]\\
&\le \mathbb{E}[X^2 \wedge \sqrt{1-r}|X|^3]\frac{1-r}{1-r^2}\\
&\le \mathbb{E}[X^2 \wedge \sqrt{1-r}|X|^3].
\end{aligned}$$

By the Dominated Convergence Theorem, the right-hand side above tends to 0 as $r \to 1$.

Now, since $|\log(1-x) - x| \le c_2 x^2$,

$$\left|\log \prod_{j=0}^{n}\left[1 - \frac{(1-r)r^{2i}t^2}{2}\right] - \sum_{i=0}^{n} \frac{(1-r)r^{2i}t^2}{2}\right| \le c_2 \sum_{i=0}^{n}\left((1-r)^2 r^{4i} t^4\right)$$
$$= c^2 \frac{t^4(1-r)^2}{1-r^4}$$
$$= c^2 t^4 (1-r)\,.$$

Thus,

$$\log \prod_{j=0}^{n}\left[1 - \frac{(1-r)r^{2i}t^2}{2}\right] = \frac{t^2(1-r^{2(n+1)})(1-r)}{2(1-r^2)} + \varepsilon'(r) = \frac{t^2(1-r^{2(n+1)})}{2(1+r)} + \varepsilon'(r),$$

where $|\varepsilon'(r)| \le c^2 t^4 (1-r)$. Thus,

$$\prod_{j=0}^{n}\left[1 - \frac{(1-r)r^{2j}t^2}{2}\right] = \exp\left(-\frac{t^2(1-r^{2(n+1)})}{2(1+r)}\right) e^{\varepsilon'(r)}\,.$$

We have

$$\left|\varphi^{(r)}(t) - e^{-t^2/4}\right| \le \left|\varphi^{(r)}(t) - \varphi_n^{(r)}(t)\right| + \left|\varphi_n^{(r)}(t) - \prod_{j=0}^{n}\left[1 - \frac{(1-r)r^{2j}t^2}{2}\right]\right|$$
$$+ \left|\prod_{j=0}^{n}\left[1 - \frac{(1-r)r^{2j}t^2}{2}\right] - e^{-t^2/[2(1+r)]+\varepsilon'(r)}\right| + \left|e^{-t^2/[2(1+r)]+\varepsilon'(r)} - e^{-t^2/4}\right|\,.$$

Thus, we can take n large enough to make the first and third terms less than ε, and then we can take r close enough to 1 to make the second term and the last term less than ε, so that the right-hand side is bounded by 4ε. We can conclude that

$$\lim_{r \to 1} \varphi^{(r)}(t) = e^{-t^2/4}\,.$$

Thus $\gamma^2 = 2$, and we have $\sqrt{1-r}\sum_{j=0}^{\infty} r^j X_j$ converges to a Normal(0, 2) distribution. ■

Problem 5. Let $\{B_t\}$ be a standard Brownian motion.

(a) Show that

$$\beta_t := \int_0^t \frac{B_s}{|B_s|} dB_s$$

is a Brownian motion.

(b) Show that $X_t = B_t^2$ satisfies

$$X_t = 2\int_0^t \sqrt{X_s} d\beta_s + t\,.$$

Solution to Problem 5. We have

$$\langle \beta \rangle_t = \int_0^t ds = t\,,$$

whence by Lévy's Theorem, $\{\beta_t\}$ is a Brownian motion.

Applying Itô's Formula,

$$\begin{aligned} B_t^2 &= 2\int_0^t B_s dB_s + t \\ &= 2\int_0^t |B_s| \frac{B_s}{|B_s|} dB_s + t \\ &= 2\int_0^t \sqrt{X_s} d\beta_s + t\,. \end{aligned}$$

■

Problem 6. Let $\{X_n\}$ be an irreducible and aperiodic Markov chain with transition matrix P. Suppose that there is a function f such that

- $Pf(x) \leq \alpha f(x)$ for all x outside a finite set of states B, where $\alpha < 1$,
- $f(x) > M$ for $x \notin B$.

Show that $\mathbb{E}_x \tau_B < \infty$, where $\tau_B = \inf\{n \geq 0 : X_n \in B\}$.

Hint: Consider the process $n \mapsto \alpha^{-n} f(X_n)$.

Solution to Problem 6. We have that if $Y_n = \alpha^{-n} f(X_n)$, then for $X_n \notin B$,

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \alpha^{-n-1} \mathbb{P} f(X_n) \leq \alpha^{-n} f(X_n) = Y_n,$$

whence $\{Y_n\}$ is a non-negative supermartingale off of B. Let $\tau = \min\{n \geq 0 : X_n \in B\}$. Then for $x \notin B$, by the Optional Stopping Theorem,

$$f(x) \geq \mathbb{E}_x Y_{\tau \wedge n} = \mathbb{E}_x[f(X_{\tau \wedge n})(1/\alpha)^{\tau \wedge n}].$$

By Fatou's Lemma,

$$\begin{aligned} f(x) &\geq \liminf_{n \to \infty} \mathbb{E}_x Y_{\tau \wedge n} \\ &\geq \mathbb{E}_x \liminf_{n \to \infty} Y_{\tau \wedge n} \\ &\geq \mathbb{E}_x \liminf_{n \to \infty} M(1/\alpha)^{\infty} \mathbf{1}\{\tau = \infty\} + \mathbb{E}_x (1/\alpha)^{\tau} \mathbf{1}\{\tau < \infty\}. \end{aligned}$$

We conclude that $\mathbb{P}(\tau = \infty) = 0$ and thus

$$\infty > f(x) \geq \mathbb{E}_x (1/\alpha)^{\tau},$$

whence in particular $\mathbb{E}_x \tau < \infty$. ■

Problem 7. Let $X_1, \ldots$ be an i.i.d. sequence of $N(0,1)$ random variables. Show that

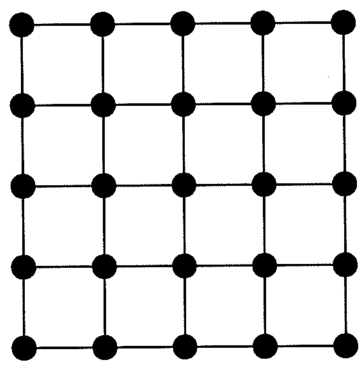
$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{n \log n}} \le \sqrt{2}.$$

Solution to Problem 7. Note that $S_n/\sqrt{n}$ has a standard Normal distribution. Let φ be the density of the standard Normal distribution, and recall that $P(N > x) \le \varphi(x)/x$, where N is a standard Normal r.v.

$$\begin{aligned}
\mathbb{P}(S_n > K\sqrt{n \log n}) &= P(N > K\sqrt{\log n}) \\
&\le \frac{\varphi(K\sqrt{\log n})}{\sqrt{K \log n}} \\
&= \frac{c_1}{\sqrt{K \log n}} e^{-\frac{1}{2}K^2 \log n} \\
&= \frac{c_1}{\sqrt{K \log n} n^{K^2/2}}
\end{aligned}$$

Thus for $K > \sqrt{2}$, the right-hand side is summable, whence $S_n/\sqrt{n \log n} < K$ eventually. ■

Problem 8. Consider the simple random walker on the two-dimensional 5×5 box shown below. If the walker starts in the lower left corner, what is the expected time until it returns to its starting position?



Solution. The stationary distribution is proportional to the degree. The degrees are given in the table below:

2	3	3	3	2
3	4	4	4	3
3	4	4	4	3
3	4	4	4	3
2	3	3	3	2

Thus the sum of the degrees equals 80, whence the stationary measure of the lower-left corner is $2/80 = 1/40$. Finally,

$$\mathbb{E}_x \tau_x = \frac{1}{\pi(x)} = 40\,.$$

■

Problem 9. Let $\{B_t\}_{t\geq 0}$ be a Brownian motion. Let $\tau_0 = \inf\{t > 0 : B_t \neq 0\}$. Show that $\mathbb{P}_0(\tau_0 > 0) = 0$.

Solution to Problem 9. Consider the event A_ε that $B_t = 0$ for some $t < \varepsilon$. By scaling,

$$\begin{aligned}\mathbb{P}_0(A_\varepsilon) = \mathbb{P}_0(B_t = 0 \text{ for some } t < \varepsilon) &= \mathbb{P}_0(a^{-1}B_{a^2 t} = 0 \text{ for some } t < \varepsilon) \\ &= \mathbb{P}_0(B_t = 0 \text{ for some } t < \varepsilon a^2) = \mathbb{P}_0(A_{a^2\varepsilon})\,.\end{aligned}$$

Letting $a \to \infty$ shows that

$$\mathbb{P}_0(A_\varepsilon) = \mathbb{P}_0(\tau_0 < \infty) = 1,$$

since one-dimensional Brownian motion is recurrent. Since this holds for all ε, we must have that 0 is an accumulation point for the zero set, whence $\mathbb{P}_0(\tau_0 = 0) = 1$. ■

Problem 10.

(a) Prove the Paley-Zygmund inequality:

$$\mathbb{P}(Z > \lambda\mathbb{E}(Z)) \geq \frac{(1-\lambda)^2 \mathbb{E}Z^2}{\mathbb{E}^2 Z}.$$

(b) Use part (a) to show the following:

Suppose that $\{A_n\}$ is a sequence of events satisfying for some $0 < c < \infty$

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty \quad \text{and} \quad \sum_{1 \leq m,n \leq N} \mathbb{P}(A_n \cap A_m) \leq c \left[\sum_{n \leq N} \mathbb{P}(A_n) \right]^2 .$$

Show that $\mathbb{P}(A_n \text{ i.o.}) > 0$.

(c) Deduce from (b) the second Borel-Cantelli lemma: if $\{A_n\}$ are independent with $\sum_n \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(A_n \text{ i.o.}) = 1$.

Solution. We have by Cauchy-Schwartz,

$$\mathbb{E}Z^2 \cdot \mathbb{P}(Z > \lambda\mathbb{E}Z) \geq \left[\mathbb{E}Z\mathbf{1}_{\{Z>\lambda\mathbb{E}Z\}}\right]^2 . \tag{4}$$

Also,

$$\mathbb{E}Z\mathbf{1}_{\{Z>\lambda\mathbb{E}Z\}} = \mathbb{E}Z - \mathbb{E}Z\mathbf{1}_{Z\leq\lambda\mathbb{E}Z} \geq \mathbb{E}Z - \lambda\mathbb{E}Z = (1-\lambda)\mathbb{E}Z . \tag{5}$$

Combining (4) with (5) yields

$$\mathbb{E}Z^2 \cdot \mathbb{P}(Z > \lambda\mathbb{E}Z) \geq (1-\lambda)^2 \mathbb{E}^2 Z .$$

Rearranging shows that

$$\mathbb{P}(Z > \lambda\mathbb{E}Z) \geq \frac{(1-\lambda)^2 \mathbb{E}^2 Z}{\mathbb{E}Z^2} .$$

Let $Z_N = \sum_{n\leq N} \mathbf{1}_{A_n}$. Then

$$\mathbb{E}Z_N^2 = \sum_{m,n\leq N} \mathbb{P}(A_n A_m) \leq c\mathbb{E}^2 Z_N .$$

Using (a) shows that

$$\mathbb{P}(Z_N \geq c\mathbb{E}Z_N) \geq \frac{(1-\lambda)^2}{c}$$

Thus, since $Z := \sum_{n=1}^{\infty} Z_n \geq Z_N$ for all N,

$$\mathbb{P}(Z \geq c\mathbb{E}Z_N) \geq P(Z_N \geq c\mathbb{E}Z_N) \geq \frac{(1-\lambda)^2}{c}.$$

Since $\{Z \geq c\mathbb{E}Z_N\} \downarrow \{Z = \infty\}$, we have

$$\mathbb{P}(A_n \text{ i.o.}) = \mathbb{P}(Z = \infty) = \lim_{n\to\infty} \mathbb{P}(Z \geq c\mathbb{E}Z_N) \geq \frac{(1-\lambda)^2}{c} > 0.$$

Finally, if $\{A_n\}$ are independent, then

$$\sum_{m,n\leq N} \mathbb{P}(A_n \cap A_m) = \sum_{m,n\leq N} \mathbb{P}(A_n)\mathbb{P}(A_m) = \sum_{m\leq N} \mathbb{P}(A_m) \sum_{n\leq N} \mathbb{P}(A_n) = \left[\sum_{m\leq N} \mathbb{P}(A_m)\right]^2$$

Thus by part (b), $\mathbb{P}(A_n \text{ i.o.}) > 0$. By the Kolmogorov zero-one law, it must be that $\mathbb{P}(A_n \text{ i.o.}) = 1$. ■