Probability Qualifying Examination
Fall 2008
University of Oregon
Department of Mathematics

STOP! Choose 8 problems from the 10 below.

Problem	Possible Points	Earned Points
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	80	

Problem 1. Prove that if X is a random variable satisfying $X \geq 0$ and $\mathbb{E}X = 0$, then $X = 0$ almost surely.

Problem 2. Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{E}(X_1) = 0$ and $\mathbb{E}(X_1^2) = 1$. Let $S_n = \sum_{k=1}^n X_k$. Prove that $n^{-1/2} S_n$ converges in distribution to the standard normal distribution.

Note: Do **not** assume any version of the Central Limit Theorem (that is what you are asked to prove!) You may assume the continuity theorem.

Problem 3. Let R_n be the length of the longest consecutive run of "heads", up to the nth coin toss, in an infinite sequence of i.i.d. fair coin tosses. Prove that $R_n / \log_2 n \to 1$ a.s. (Here $\log_2 n$ denotes the logarithm of n in base 2.)

Hint: For the upper bound, consider first the sequence $n_k = e^k$.

Problem 4. Let p be a transition matrix for an irreducible Markov chain $\{Y_n\}$ with state-space $\mathcal{X}$ and let f be a function $f : \mathcal{X} \to \mathbb{R}$ satisfying $pf = f$, meaning that

$$\sum_{y \in \mathcal{X}} f(y) p(x, y) = f(x)$$

for all $x \in \mathcal{X}$. Show that if

$$\sum_{n=0}^{\infty} \mathbb{E}[(f(Y_{n+1}) - f(Y_n))^2] < \infty,$$

then $f(Y_n)$ converges almost surely.

Problem 5. Let $\{X_t\}$ be a simple random walk on $\{0, 1, 2 \ldots, n\}$ with absorption at 0 and n. Let τ be the time of absorption. Show that

$$\mathbb{E}_k[\tau \mid X_\tau = n] = \frac{n^2 - k^2}{3}.$$

Hint: Consider the process $Z_t = X_t^3 - 3tX_t$. What kind of a process is $\{Z_t\}$?

Problem 6. Let $\{B_t\}$ be a Brownian motion started at 0. Find the distribution of

$$\tau_1 := \inf\{t \geq 0 : B_t = 1\}.$$

Problem 7. Show that *geometric Brownian motion*, $X_t = e^{(\mu - \sigma^2/2)t + \sigma B_t}$ satisfies the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dB_t,$$

where $\{B_t\}$ is a Brownian motion.

Problem 8. Show that the set of zeros of Brownian motion is unbounded, a.s.

Problem 9. Suppose that $\{X_n\}$ is a sequence of L^2-random variables with $\mathbb{E}X_n = 0$ for all n and suppose that there is a constant c_1 such that

$$\mathrm{Var}(X_{n+k} - X_n) \leq c_1 k \quad \text{for all } n, k \geq 0.$$

Prove that $n^{-1}X_n \to 0$ almost surely.
Hint: First prove along a suitably chosen subsequence.

Problem 10. Let $\{X_n\}_{n=0}^{\infty}$ be the Markov chain on $\mathbb{Z}^+ = \{0, 1, \ldots\}$ with transition matrix

$$P(k, j) = \begin{cases} p & \text{if } j = k + 1, \\ q & \text{if } j = k - 1 \text{ and } k \neq 0, \text{ or if } k = j = 0. \end{cases}$$

Here $q = 1 - p$, and we assume that $p < q$. In words, this is the nearest-neighbor walk on $\mathbb{Z}$ which censors any move below 0. Let $\tau_0^+ = \inf\{n > 0 : X_n = 0\}$. Find $\mathbb{E}_0\tau_0^+$.

Solutions

Solution to Problem 1. Suppose that $\mathbb{P}\{X > 0\} > 0$. Consider the events $A_n = \{X > 1/n\}$. Then $A_n \supset A_{n+1}$, and $A_n \downarrow \{X > 0\}$. Consequently $\mathbb{P}\{X > 0\} = \lim_{n\to\infty} \mathbb{P}\{X > 1/n\}$, and so for n large enough, $\mathbb{P}\{X > 1/n\} > 0$. Then

$$\int X d\mathbb{P} \geq \int_{A_n} X d\mathbb{P} \geq \frac{1}{n}\mathbb{P}\{X > 1/n\} > 0.$$

■

Solution to Problem 2. Let φ be the characteristic function of X, and observe that the ch.f. of $n^{-1/2}S_n$ is $\varphi(t/\sqrt{n})^n$.

Expanding $e^{i\theta}$ out to two terms and using the Lagrangian form for the remainder shows that

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2} - \frac{ie^{i\theta^\star}\theta^3}{6}, \tag{1}$$

for some $\theta^\star \in [0, \theta]$. The final term is bounded in absolute value by $|\theta|^3/6$. Expanding to one term shows that

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \frac{e^{i\tilde{\theta}}\theta^2}{2} \\ &= 1 + i\theta - \frac{\theta^2}{2} - \frac{[e^{i\tilde{\theta}} - 1]\theta^2}{2}. \end{aligned} \tag{2}$$

for some $\tilde{\theta} \in [0, \theta]$. The last term is bounded by θ^2. Together the expansions (1) and (2), along with the explicit bounds on the remainder terms show that

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2} + r(\theta),$$

where $|r(\theta)| \leq |\theta|^3 \wedge \theta^2$. Therefore,

$$\begin{aligned} \varphi(t/\sqrt{n}) &= 1 + it\frac{\mathbb{E}X}{\sqrt{n}} - \frac{t^2\mathbb{E}X^2}{2n} + \mathbb{E}r(tX/\sqrt{n}) \\ &= 1 - \frac{t^2}{2n} + \mathbb{E}r(tX/\sqrt{n}). \end{aligned}$$

Note that

$$n|\mathbb{E}r(t/\sqrt{n}| \leq n\mathbb{E}|r(t/\sqrt{n})| \leq n\mathbb{E}\left[\frac{t^2X^2}{n} \wedge \frac{t^3|X|^3}{n^{3/2}}\right] = \mathbb{E}\left[t^2X^2 \wedge \frac{t^3|X|^3}{\sqrt{n}}\right].$$

The last term tends to 0 as $n \to \infty$, by the Dominated Convergence Theorem. Therefore,

$$\varphi(t/\sqrt{n}) = 1 - \frac{t^2}{2n} + o(n^{-1}).$$

Using the inequality $|z^n - w^n| \le n|z - w|$ valid for complex z, w in the unit ball around 0,

$$|\varphi(t/\sqrt{n})^n - (1 - t^2/(2n))^n| \le no(n^{-1}) \to 0.$$

But clearly $(1 - t^2/(2n))^n \to e^{-t^2/2}$, whence $\varphi(n^{-1/2}t)^n \to e^{-t^2/2}$. By the continuity theorem, the result is proven. ■

Solution to Problem 3. First the lower bound: Consider the $n/(1-\epsilon)\log_2 n$ disjoint blocks of length $(1-\epsilon)\log_2 n$ before time n. The chance that there is at least one tail in a block equals

$$p_n = 1 - 2^{-(1-\epsilon)\log_2 n} = 1 - n^{-(1-\epsilon)}.$$

Thus, the chance that every block has at least one tail equals

$$p_n^{n/[(1-\epsilon)\log_2 n]} = (1 - n^{-(1-\epsilon)})^{n/[(1-\epsilon)\log_2 n]} \le e^{-n^\epsilon/[(1-\epsilon)\log_2 n]}.$$

The right-hand side is summable, so the Borel-Cantelli implies that only finitely often do all blocks contain at least one tail. Thus $R_n > (1-\epsilon)\log_2 n$ eventually.

For the upper bound, by considering all (some overlapping) blocks of length $(1+\epsilon)\log_2 n$ before time n

$$\mathbb{P}\{R_n > (1+\epsilon)\log_2 n\} = \mathbb{P}\left(\bigcup_{\text{blocks}} \{\text{block contains no tail}\}\right) \le n2^{-(1+\epsilon)\log_2 n} = n^{-\epsilon}.$$

Then along the subsequence $n_k = e^k$, by applying Borel-Cantelli, $R_{n_k} \le (1+\epsilon)\log_2 n_k$ a.s. Note that

$$\log_2 e^{k+1} = \log_2 e^k + \log_2 e = (\log_2 e^k)[1 + \log_2/\log_2 e^k] = (\log_2 e^k)[1 + o(1)],$$

whence for k large, $\log_2 e^{k+1} \le (1+\epsilon)\log e^k$. Thus, for $n_k \le n \le n_{k+1}$, if n is large enough,

$$R_n \le R_{n_{k+1}} \le (1+\epsilon)\log_2 e^{k+1} \le (1+\epsilon)^2\log_2 e^k \le (1+\epsilon)^2\log_2 n.$$

Since ϵ was arbitrary, this proves the upper bound.

■

Solution to Problem 4. Let $M_n = f(Y_n)$. The condition on f implies that $\{M_n\}$ is a martingale. By the orthogonality of martingale increments,

$$\mathbb{E}M_n^2 = \mathbb{E}M_0^2 + \sum_{k=0}^{n-1} \mathbb{E}(M_{k+1} - M_k)^2 \leq \mathbb{E}M_0^2 + \sum_{k=0}^{\infty} \mathbb{E}(M_{k+1} - M_k)^2 = B < \infty.$$

Therefore, M_n is bounded in L^2. By the Martingale Convergence Theorem, M_n must converge almost surely. ■

Solution to Problem 5. Note that $Z_{t\wedge\tau}$ is a martingale: Let $\Delta_t = X_{t+1} - X_t$. If $t < \tau$, then

$$\begin{aligned}
\mathbb{E}[Z_{t+1} \mid \mathcal{F}_t] &= \mathbb{E}[(X_t + \Delta_t)^3 - 3(t+1)(X_t + \Delta_t) \mid \mathcal{F}_t] \\
&= \mathbb{E}[X_t^3 + 3X_t^2\Delta_t + 3X_t\Delta_t^2 + \Delta_t^3 - 3tX_t - 3t\Delta_t - 3X_t - 3\Delta_t \mid \mathcal{F}_t] \\
&= X_t^3 + 3X_t^2\mathbb{E}[\Delta_t \mid \mathcal{F}_t] + 3X_t\mathbb{E}[\Delta_t^2 \mid \mathcal{F}_t] + \mathbb{E}[\Delta_t^3 \mid \mathcal{F}_t] - 3tX_t - 3t\mathbb{E}[\Delta_t] - 3X_t - 3\mathbb{E}[\Delta_t \mid \mathcal{F}_t]
\end{aligned}$$

Since $\Delta_t^2 = 1, \Delta_t^3 = \Delta_t$ and $\mathbb{E}[\Delta_t \mid \mathcal{F}_t] = 0$, we have

$$\mathbb{E}[Z_{t+1} \mid \mathcal{F}_t] = X_t^3 - 3tX_t = Z_t.$$

By Optional Stopping, and since $\mathbb{P}_k\{Z_\tau = n\} = k/n$,

$$k^3 = \mathbb{E}_k Z_\tau = \mathbb{E}_k[Z_\tau \mid X_\tau = n]\frac{k}{n} + \mathbb{E}_k[Z_\tau \mid X_\tau = 0]\frac{n-k}{n} = \frac{(n^3 - 3n\mathbb{E}[\tau \mid X_\tau = n])k}{n}.$$

Solving for $\mathbb{E}[\tau \mid X_\tau = n]$ yields the desired identity. ■

Solution to Problem 6. Applying the Strong Markov Property at τ_λ and reflecting the Brownian motion around the line $y = \lambda$ from time τ_λ onwards shows that

$$\begin{aligned}
\mathbb{P}\{B_t \geq 1\} &= \mathbb{P}\{\tau_1 < t,\ B_t \geq 1\} \\
&= \mathbb{P}\{\tau_1 < t, B_{t-\tau_1} - B_{\tau_1} > 0\} \\
&= \frac{1}{2}\mathbb{P}\{\tau_1 < t\}.
\end{aligned}$$

Thus,

$$\mathbb{P}\{\tau_1 < t\} = 2\mathbb{P}\{B_t/\sqrt{t} \geq 1/\sqrt{t}\} = \int_{1/\sqrt{t}} \varphi(u)du,$$

where $\varphi(u) = (2\pi)^{-1/2}\exp(-u^2/2)$ is standard Gaussian density. Differentiating, the density of τ_1 is

$$\frac{\varphi(1/\sqrt{t})}{2\sqrt{t}}, \quad t \geq 0.$$

■

Solution to Problem 7. Applying Itô's formula to the function $f(t,b) = e^{(\mu-\sigma^2/2)t+\sigma b}$, we have

$$\begin{aligned} df(t,B_t) &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial b}dB_t + \frac{1}{2}\frac{\partial f^2}{\partial b^2}dt \\ &= \left(\frac{\partial f}{\partial t} + \frac{\partial f^2}{\partial b^2}\right)dt + \frac{\partial f}{\partial b}dB_t \\ &= \mu X_t dt + \sigma X_t dB_t. \end{aligned}$$

■

Solution to Problem 8. Note that by scaling,

$$\mathbb{P}\{B_t > k\sqrt{t}\} = \mathbb{P}\{B_1 > k\} > 0$$

Thus

$$\begin{aligned} \mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{m\geq n}\{B_m > k\sqrt{m}\}\right) &= \lim_{n\to\infty}\mathbb{P}\left(\bigcup_{m\geq n}\{B_m > k\sqrt{m}\}\right) \\ &\geq \lim_{n\to\infty}\sup_{m\geq n}\mathbb{P}\{B_m > k\sqrt{m}\} = \limsup_{n\to\infty}\mathbb{P}\{B_n > k\sqrt{n}\} = \mathbb{P}\{B_1 > k\} > 0. \end{aligned}$$

Thus $B_n/\sqrt{n} > k$ i.o., for all k, a.s. Since $\{-B_t\}$ has the same distribution as $\{B_t\}$, we conclude that $B_n/\sqrt{n} < -k$ i.o., for all k, a.s. Therefore, it is true that

$$\limsup_{n\to\infty} B_n = \infty, \quad \liminf_{n\to\infty} B_n = -\infty.$$

By the mean-value theorem and continuity of Brownian motion, it must be that $B_n = 0$ for infinitely many n. ■

Solution to Problem 9. Note that by Chebshev's inequality,

$$\mathbb{P}\{|X_{n^2}| > \epsilon n^2\} \leq \frac{\text{Var}(X_n)}{\epsilon^2 n^4} \leq \frac{c_1}{\epsilon^2 n^2}.$$

Since the right-hand side above is summable, the Borel-Cantelli Lemma implies that $n^{-2}X_{n^2} < \epsilon$ eventually, a.s. Since ϵ was arbitrary, $n^{-2}X_{n^2} \to 0$ a.s.

Also,

$$\mathbb{P}\left\{\sup_{n^2<k\le(n+1)^2}|X_k - X_{n^2}| > \epsilon n^2\right\} \le \sum_{k=n^2+1}^{(n+1)^2} \mathbb{P}\{|X_k - X_{n^2}| > \epsilon n^2\}$$

$$\le \sum_{k=n^2+1}^{(n+1)^2} \frac{\mathrm{Var}(X_k - X_{n^2})}{\epsilon^2 n^4}$$

$$\le \sum_{k=n^2+1}^{(n+1)^2} \frac{c_1(k-n^2)}{\epsilon^2 n^4}$$

$$\le \frac{c_1(2n)^2}{\epsilon^2 n^4}$$

$$\le \frac{c_2}{\epsilon^2 n^2}$$

Since the right-hand side is summable, Borel-Cantelli implies that

$$\sup_{n<k\le(n+1)^2}|X_k - X_{n^2}| \le \epsilon n^2 \quad \text{for } n \text{ large enough,} \quad \text{a.s.}$$

Therefore, if $n^2 < k \le (n+1)^2$, then

$$\frac{X_k}{k} \le \frac{X_{n^2} + X_k - X_{n^2}}{n^2} \le 2\epsilon$$

if n is large enough. Similarly,

$$\frac{X_k}{k} \ge \frac{X_{n^2} + X_k - X_{n^2}}{(n+1)^2} \ge \frac{n^2}{(n+1)^2}\frac{X_{n^2}}{n^2} - \frac{X_k - X_n^2}{n^2} \ge -2\epsilon,$$

for n large enough. We conclude that $k^{-1}X_k \to 0$ a.s. ■

Solution to Problem 10. We first show the chain is positive recurrent by finding a stationary probability measure, solving for π in the system of equations $\pi = \pi P$

We have

$$\pi(0) = q\pi(1) + q\pi(0)$$
$$\pi(1) = \left(\frac{p}{q}\right)\pi(0),$$

and for $k > 0$,

$$\pi(k) = q\pi(k+1) + p\pi(k).$$

It is seen inductively that $\pi(k) = (p/q)^k \pi(0)$ is a solution. We have that

$$\sum_{k=0}^{\infty} \pi(0)(p/q)^k = \frac{\pi(0)}{1 - (p/q)},$$

whence taking $\pi(0) = 1 - (p/q)$ yields a probability distribution on $\mathbb{Z}^+$. This is the unique stationary distribution for P, which implies that

$$\mathbb{E}_0 \tau_0^+ = \frac{1}{\pi(0)} = \frac{1}{1 - p/q}.$$

■