Probability Qualifying Examination

September, 2006

Student Name: ______________________ **Student Number:** ___________

Note:
(1) Your solution to each question should be legible and include enough details to present your ideas. No details will have no scores.
(2) This examination has total 12 questions. Points for each question is marked at the beginning of the question (total 120 points).

The following table is for office use only

Question Number	Points Obtained	Points
Q1		10
Q2		10
Q3		10
Q4		10
Q5		10
Q6		10
Q7		10
Q8		10
Q9		10
Q10		10
Q11		10
Q12		10
Total Points		120

(1a) (5 points) We assume that the filtration $\mathcal{F}_t$ is right continuous. Prove that σ is a stopping time if and only if $\{\sigma < t\} \in \mathcal{F}_t$ for every $t \geq 0$.
(1b)(5 points) Suppose that $\{X_t, t \geq 0\}$ is a $\mathcal{F}_t$-martingale. Prove that $\{|X_t|, t \geq 0\}$ is a submartingale.

(2) (10 points) Suppose that X, Y are discrete random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and there exist two index sets Γ_1, Γ_2, where $\{\Gamma_i, i = 1, 2\}$ are finite or countable infinite sets, and $\{x_n \in \mathbb{R}, n \in \Gamma_1\}$, $\{y_m \in \mathbb{R}, m \in \Gamma_2\}$ such that $p_n = \mathbb{P}(X = x_n) > 0$, $q_m = \mathbb{P}(Y = y_m) > 0$, $\sum_{n\in\Gamma_1} p_n = 1 = \sum_{m\in\Gamma_2} q_m$. Let $\mathbb{E}(|Y|) < \infty$. Prove that
(a) There exists $g_r \in \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ such that

$$\mathbb{E}(Y|X) = g_r(X);$$

(b)

$$g_r(x_n) = \mathbb{E}(Y|X = x_n) = \sum_{m\in\Gamma_2} \mathbb{P}(Y = y_m|X = x_n) y_m.$$

(3) (10 points) Let $T = \{0, 1, 2, \cdots\}$ and $\{X_n, n \in T\}$ be a supermartingale relative to $\{\mathcal{F}_n, n \in T\}$ and let $\{\tau_k, k \in T\}$ be a system of bounded, discrete stopping times such that $\tau_k \leq \tau_{k+1}$ for all ω. Define $Y_k \triangleq X_{\tau_k}$ and $\mathcal{G}_k \triangleq \mathcal{F}_{\tau_k}$. Prove that $\{Y_k, k \geq 0\}$ is a $\mathcal{G}_k$-supermartingale.

(4) (10 points) Let $T = \{0, 1, 2, \cdots\}$ and $\{X_n, n \in T\}$ be a martingale relative to $\{\mathcal{F}_n, n \in T\}$ such that $\mathbb{E}(|X_n|^p) < \infty, n = 0, 1, 2, \cdots$ for some $p \geq 1$. Prove that
(A) For every N, we have

$$\mathbb{P}(\max_{0\leq n\leq N} |X_n| \geq \lambda) \leq \mathbb{E}(|X_N|^p)/\lambda^p, \quad \lambda > 0.$$

(B) For every N, if $p > 1$,

$$\mathbb{E}\{\max_{0\leq n\leq N} |X_n|^p\} \leq (\frac{p}{p-1})^p \cdot \mathbb{E}(|X_N|^p).$$

(5) (10 points) Let $(U, \mathcal{F}_U)$,$(V, \mathcal{F}_V)$ and $(W, \mathcal{F}_W)$ be three measure spaces, $F(u, w) : U \times W \to \mathbb{R}$ be a $\mathcal{F}_U \times \mathcal{F}_W$-measurable function. Suppose that for each $v \in V$, $\mathbb{P}_v(\cdot)$ is a measure on $\mathcal{F}_W$ and for each fixed $\Gamma \in \mathcal{F}_W$, $\mathbb{P}_\cdot(\Gamma) \in \mathcal{F}_V$. If for each $(u, v) \in U \times V$ the integral

$$G(u, v) = \int_W F(u, w)\mathbb{P}_v(dw)$$

is finite, prove that $G \in \mathcal{F}_U \times \mathcal{F}_V$.

(6) (10 points) Suppose that X is a nonnegative but not necessarily integrable random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$. Prove

$$\mathbb{E}(X|\mathcal{G}) = \int_0^\infty \mathbb{P}(X > t|\mathcal{G})dt. \tag{0.1}$$

(7) (10 points) Let $\{X_n, n \geq 0\}$ be a time homogeneous Markov process with state space $E = \{0, 1, 2, 3, \cdots\}$. Define

$$T_k = \inf\{n \geq 1 : X_n = k\}.$$

Prove that

$$\mathbb{E}^a\{\sum_{n=0}^{T_k-1} 1_{\{X_n=j\}}\} = \mathbb{E}^a\{\sum_{n=0}^{\infty} 1_{\{X_n=j, n\leq T_k-1\}}\},$$

where $a \in E$ is the initial state.

(8) (10 points) (Basic Lemma) Let (S, ρ) be a metric space and $\{\xi_n, n \in \mathbb{N}\}$ be a S-valued random variables. Then, $\xi_n \to \xi$ in probability if and only if every subsequence $\{\xi_{n'}, n' \in \mathbb{N}' \subset \mathbb{N}\}$ has a further subsequence $\{\xi_{n''}, n'' \in \mathbb{N}'' \subset \mathbb{N}'\}$ such that $\xi_{n''} \to \xi$ a.s. as n'' along $\mathbb{N}''$ goes to infinity.

(9) (10 points) For any metric spaces S and T, let $\xi, \xi_1, \xi_2, \cdots$ be random elements in S with $\xi_n \to \xi$ in probability, and let the mapping $f : S \to T$ be measurable and *a.s.* continuous at ξ. Prove that $f(\xi_n) \to f(\xi)$ in probability.

(10) (10 points) Let $\xi, \xi_1, \xi_2, \cdots$ be random elements in a metric space (S, ρ). Prove that $\xi_n \to \xi$ in probability implies $\xi_n \to \xi$ in distribution, and the two conditions are equivalent when ξ is *a.s.* constant.

(11) (10 points) Let $\{B_t\}$ be a Brownian motion. Show that almost surely,

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log t}} \leq 1.$$

(12) (10 points) Let $\{B_t\}$ be Brownian motion in $\mathbb{R}^2$. Let

$$\tau_a = \inf\{t > 0 \,:\, |B_t| = a\}.$$

Suppose that $a < |x| < b$. Show that

$$\mathbb{P}_x\{\tau_b < \tau_a\} = \frac{\log(|x|^2) - \log(|a|^2)}{\log(|b|^2) - \log(|a|^2)}.$$

Probability Qualifying Examination Key

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Note:
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Total Points		120

(1a) (5 points) We assume that the filtration $\mathcal{F}_t$ is right continuous. Prove that σ is a stopping time if and only if $\{\sigma < t\} \in \mathcal{F}_t$ for every $t \geq 0$.
(1b)(5 points) Suppose that $\{X_t, t \geq 0\}$ is a $\mathcal{F}_t$-martingale. Prove that $\{|X_t|, t \geq 0\}$ is a submartingale.
Proof. (1a) If σ is a stopping time, then $\{\sigma < t\} = \cup_n \{\sigma \leq t - 1/n\} \in \mathcal{F}_t$. Conversely, if $\{\sigma < t\} \in \mathcal{F}_t$ for every $t \geq 0$, then $\{\sigma \leq t\} = \cap_n \{\sigma < t + 1/n\} \in \mathcal{F}_{t+0} = \mathcal{F}_t$.
(1b) For any $0 \leq s \leq t$, since $\{X_t, t \geq 0\}$ is a $\mathcal{F}_t$-martingale, then

$$\mathbb{E}(X_t|\mathcal{F}_s) = X_s.$$

So, $|\mathbb{E}(X_t|\mathcal{F}_s)| = |X_s|$. By Jensen's inequality, we have

$$|\mathbb{E}(X_t|\mathcal{F}_s)| \leq \mathbb{E}(|X_t||\mathcal{F}_s).$$

Together, we get

$$|X_s| = |\mathbb{E}(X_t|\mathcal{F}_s)| \leq \mathbb{E}(|X_t||\mathcal{F}_s).$$

(2) (10 points) Suppose that X, Y are discrete random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and there exist two index sets Γ_1, Γ_2, where $\{\Gamma_i, i = 1, 2\}$ are finite or countable infinite sets, and $\{x_n \in \mathbb{R}, n \in \Gamma_1\}$, $\{y_m \in \mathbb{R}, m \in \Gamma_2\}$ such that $p_n = \mathbb{P}(X = x_n) > 0$, $q_m = \mathbb{P}(Y = y_m) > 0$, $\sum_{n\in\Gamma_1} p_n = 1 = \sum_{m\in\Gamma_2} q_m$. Let $\mathbb{E}(|Y|) < \infty$. Prove that
(a) There exists $g_r \in \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ such that

$$\mathbb{E}(Y|X) = g_r(X);$$

(b)

$$g_r(x_n) = \mathbb{E}(Y|X = x_n) = \sum_{m\in\Gamma_2} \mathbb{P}(Y = y_m|X = x_n) y_m.$$

Proof. (a) The existence of g_r follows from Doob's composition theorem.
(b) From part (a), we get $g_r(x_n) = \mathbb{E}(Y|x_n)$. Since

$$\begin{aligned}
\int_{\{X=x_n\}} \mathbb{E}(Y|X = x_n) d\mathbb{P} &= \int_{\{X=x_n\}} Y d\mathbb{P} \\
&= \sum_{m\in\Gamma_2} y_m \mathbb{P}(X = x_n, Y = y_m) \\
&= \sum_{m\in\Gamma_2} y_m \frac{\mathbb{P}(X = x_n, Y = y_m)}{\mathbb{P}(X = x_n)} \mathbb{P}(X = x_n) \\
&= \sum_{m\in\Gamma_2} y_m \int_{\{X=x_n\}} \mathbb{P}(Y = y_m|X = x_n) d\mathbb{P} \\
&= \int_{\{X=x_n\}} \sum_{m\in\Gamma_2} y_m \mathbb{P}(Y = y_m|X = x_n) d\mathbb{P},
\end{aligned}$$

this proves that $\mathbb{E}(Y|X=x_n) = \sum_{m\in\Gamma_2} y_m \mathbb{P}(Y=y_m|X=x_n)$.

(3) (10 points) Let $T=\{0,1,2,\cdots\}$ and $\{X_n, n\in T\}$ be a supermartingale relative to $\{\mathcal{F}_n, n\in T\}$ and let $\{\tau_k, k\in T\}$ be a system of bounded, discrete stopping times such that $\tau_k \le \tau_{k+1}$ for all ω. Define $Y_k \triangleq X_{\tau_k}$ and $\mathcal{G}_k \triangleq \mathcal{F}_{\tau_k}$. Prove that $\{Y_k, k\ge 0\}$ is a $\mathcal{G}_k$-supermartingale.

Proof. We only need to prove that for each k

$$\mathbb{E}(Y_k|\mathcal{G}_{k-1}) \le Y_{k-1}$$

or

$$\mathbb{E}(X_{\tau_k}|\mathcal{F}_{\tau_{k-1}}) \le X_{\tau_{k-1}}. \tag{0.1}$$

Since τ_k is bounded, let $m\in T$ be such that $\tau_k(\omega)\le m$ for all $\omega\in\Omega$. Set $f_n = 1_{\{\tau_{k-1}<n\le\tau_k\}} = 1_{\{n\le\tau_k\}} - 1_{\{n\le\tau_{k-1}\}} \ge 0$. Define $Y_0 = X_0$, $Y_n = Y_{n-1} + f_n(X_n - X_{n-1})$, $n = 1,2,\cdots$. By induction, we can get $Y_n = X_{\tau_k\wedge n} - X_{\tau_{k-1}\wedge n}$. In particular, $Y_m - X_0 = X_{\tau_k} - X_{\tau_{k-1}}$. Since

$$\begin{aligned}
\mathbb{E}(Y_k|\mathcal{F}_{k-1}) &= \mathbb{E}(Y_{k-1} + f_k(X_k - X_{k-1})|\mathcal{F}_{k-1}) \\
&= Y_{k-1} + f_k\{\mathbb{E}(X_k|\mathcal{F}_{k-1}) - X_{k-1}\} \\
&\le Y_{k-1}, \qquad (\{X_k\} \text{ is a supermartingale})
\end{aligned}$$

we get $\mathbb{E}Y_k \le \mathbb{E}Y_{k-1}$. This gives $\mathbb{E}Y_m \le \mathbb{E}X_0$ and

$$\mathbb{E}X_{\tau_k} \le \mathbb{E}X_{\tau_{k-1}}. \tag{0.2}$$

To prove (0.1), if suffices to show that for any $B\in\mathcal{F}_{\tau_{k-1}}$, we have

$$\int_B X_{\tau_k} d\mathbb{P} \le \int_B X_{\tau_{k-1}} d\mathbb{P}. \tag{0.3}$$

Set

$$\tau_{k-1}^B(\omega) = \begin{cases} \tau_{k-1}, & \text{if } \omega\in B, \\ m, & \text{if } \omega\in B^c, \end{cases}$$

and

$$\tau_k^B(\omega) = \begin{cases} \tau_k, & \text{if } \omega\in B, \\ m, & \text{if } \omega\in B^c. \end{cases}$$

Then

$$\{\tau_{k-1}^B \le j\} = \begin{cases} \Omega, & \text{if } j\ge m, \\ \{\tau_{k-1}\le j\}\cap B \in \mathcal{F}_j, & \text{if } j<m, \end{cases}$$

and

$$\{\tau_k^B \le j\} = \begin{cases} \Omega, & \text{if } j \ge m, \\ \{\tau_k \le j\} \cap B \in \mathcal{F}_j, & \text{if } j < m, \end{cases}.$$

So τ_{k-1}^B and τ_k^B are stopping times and (0.3) follows from (0.2).

(4) (10 points) Let $T = \{0, 1, 2, \cdots\}$ and $\{X_n, n \in T\}$ be a martingale relative to $\{\mathcal{F}_n, n \in T\}$ such that $\mathbb{E}(|X_n|^p) < \infty, n = 0, 1, 2, \cdots$ for some $p \ge 1$. Prove that
(A) For every N, we have

$$\mathbb{P}(\max_{0 \le n \le N} |X_n| \ge \lambda) \le \mathbb{E}(|X_N|^p)/\lambda^p, \quad \lambda > 0.$$

(B) For every N, if $p > 1$,

$$\mathbb{E}\{\max_{0 \le n \le N} |X_n|^p\} \le (\frac{p}{p-1})^p \cdot \mathbb{E}(|X_N|^p).$$

Proof. (A) For $\lambda > 0$, set

$$\sigma = \begin{cases} \min\{n \le N, |X_n|^p \ge \lambda^p\}, & \text{if } \{n \le N, |X_n|^p \ge \lambda^p\} \ne \emptyset, \\ N, & \text{if } \{n \le N, |X_n|^p \ge \lambda^p\} = \emptyset. \end{cases}$$

For $p \ge 1$, by Jensen's inequality, $\{|X_n|^p, n = 1, 2, \cdots\}$ is a $\mathcal{F}_n$-submartingale. Similar to the arguments in the question (3) we can get $\mathbb{E}|X_N|^p \ge \mathbb{E}|X_\sigma|^p$. Since

$$\begin{aligned} \mathbb{E}|X_\sigma|^p &= \int_{\{\max_{0\le n\le N} |X_n|^p \ge \lambda^p\}} |X_\sigma|^p d\mathbb{P} + \int_{\{\max_{0\le n\le N} |X_n|^p < \lambda^p\}} |X_N|^p d\mathbb{P} \\ &\ge \lambda^p \mathbb{P}(\max_{0\le n\le N} |X_n|^p \ge \lambda^p) + \int_{\{\max_{0\le n\le N} |X_n|^p < \lambda^p\}} |X_N|^p d\mathbb{P}, \end{aligned}$$

we get

$$\begin{aligned} \lambda^p \mathbb{P}(\max_{0\le n\le N} |X_n|^p \ge \lambda^p) &\le \mathbb{E}|X_\sigma|^p - \int_{\{\max_{0\le n\le N} |X_n|^p < \lambda^p\}} |X_N|^p d\mathbb{P} \\ &\le \mathbb{E}|X_N|^p - \int_{\{\max_{0\le n\le N} |X_n|^p < \lambda^p\}} |X_N|^p d\mathbb{P} \\ &= \int_{\{\max_{0\le n\le N} |X_n|^p \ge \lambda^p\}} |X_N|^p d\mathbb{P} \\ &\le \mathbb{E}|X_N|^p. \end{aligned}$$

This gives

$$\begin{aligned} \mathbb{P}(\max_{0\le n\le N} |X_n| \ge \lambda) &= \mathbb{P}(\max_{0\le n\le N} |X_n|^p \ge \lambda^p) \\ &\le \mathbb{E}(|X_N|^p/\lambda^p) \quad \text{for } p \ge 1. \end{aligned}$$

(A) is proved.
For part (B), set $Y = \max_{0\le n\le N}|X_n|$. From the proof of (A) we get

$$\lambda\mathbb{P}(Y \ge \lambda) \le \int_{\{Y\ge\lambda\}} |X_N| d\mathbb{P}.$$

Thus, we have

$$\begin{aligned}
\mathbb{E}Y^p &= \int_\Omega d\mathbb{P}\int_0^Y p\lambda^{p-1}d\lambda = \int_\Omega d\mathbb{P}\int_0^\infty 1_{\{Y\ge\lambda\}}p\lambda^{p-1}d\lambda \\
&= p\int_0^\infty \lambda^{p-1}\mathbb{P}(Y\ge\lambda)d\lambda \\
&\le p\int_0^\infty\int_\Omega \lambda^{p-2}1_{\{Y\ge\lambda\}}|X_N|d\mathbb{P}d\lambda \\
&= (\frac{p}{p-1})\int_\Omega Y^{p-1}|X_N|d\mathbb{P} \\
&\le (\frac{p}{p-1})\{\mathbb{E}(|X_N|^p)\}^{1/p}\{\mathbb{E}(Y^{(p-1)(\frac{p}{p-1})})\}^{\frac{p-1}{p}} \quad \text{(Hölder's inequality)} \\
&= (\frac{p}{p-1})\{\mathbb{E}(|X_N|^p)\}^{1/p}\{\mathbb{E}(Y^p)\}^{\frac{p-1}{p}} \\
&= (\frac{p}{p-1})\{\mathbb{E}(|X_N|^p)\}^{1/p}\{\mathbb{E}(Y^p)\}\{\mathbb{E}(Y^p)\}^{-1/p}
\end{aligned}$$

and

$$\{\mathbb{E}(Y^p)\}^{1/p} \le (\frac{p}{p-1})\{\mathbb{E}(|X_N|^p)\}^{1/p}.$$

This gives

$$\mathbb{E}(Y^p) \le (\frac{p}{p-1})^p\mathbb{E}(|X_N|^p).$$

(5) (10 points) Let $(U,\mathcal{F}_U)$,$(V,\mathcal{F}_V)$ and $(W,\mathcal{F}_W)$ be three measure spaces, $F(u,w): U\times W \to \mathbb{R}$ be a $\mathcal{F}_U\times\mathcal{F}_W$-measurable function. Suppose that for each $v\in V$, $\mathbb{P}_v(\cdot)$ is a measure on $\mathcal{F}_W$ and for each fixed $\Gamma\in\mathcal{F}_W$, $\mathbb{P}_\cdot(\Gamma)\in\mathcal{F}_V$. If for each $(u,v)\in U\times V$ the integral

$$G(u,v) = \int_W F(u,w)\mathbb{P}_v(dw)$$

is finite, prove that $G\in\mathcal{F}_U\times\mathcal{F}_V$.
Proof. Define

$$\begin{aligned}
\mathcal{L} = \{F(u,w): \quad & F\in\mathcal{F}_U\times\mathcal{F}_W \text{ and for any}(u,v)\in U\times V, \\
& |G(u,v)| = |\int_W F(u,w)\mathbb{P}_v(dw)| < \infty\},
\end{aligned}$$

$$\mathcal{L} = \{F(\cdot,\cdot) \in \mathcal{F}_U \times \mathcal{F}_W : G(u,v) = \int_W F(u,w)\mathbb{P}_v(dw) \in \mathcal{F}_U \times \mathcal{F}_V\},$$

$$\Pi = \{A \times B : A \in \mathcal{F}_U, B \in \mathcal{F}_W\}.$$

If $f_n \in \mathcal{L}$, $0 \le f_n \uparrow f$ and $f \in \mathcal{L}$, then by dominated convergence theorem, $f \in \mathcal{L}$. It is easy to see $1_{A\times B} \in \mathcal{L}$ for any $A \in \mathcal{F}_U$ and $B \in \mathcal{F}_W$. By $\mathcal{L}$-system method, for any $F \in \mathcal{L}$ we have $F \in \mathcal{L}$ or $G \in \mathcal{F}_U \times \mathcal{F}_V$.

(6) (10 points) Suppose that X is a nonnegative but not necessarily integrable random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$. Prove

$$\mathbb{E}(X|\mathcal{G}) = \int_0^\infty \mathbb{P}(X > t|\mathcal{G})dt. \tag{0.4}$$

Proof. Note that Fubini's theorem works for integrable f or nonnegative f. To prove (0.4), it is equivalent to showing that for any $A \in \mathcal{G}$, we have

$$\int_A X d\mathbb{P} = \int_A \int_0^\infty \mathbb{P}(X > t|\mathcal{G})dtd\mathbb{P}. \tag{0.5}$$

Since

$$\begin{aligned}
\int_A X d\mathbb{P} &= \int_A \int_0^X dtd\mathbb{P} \\
&= \int_A \int_0^\infty 1_{\{t<X\}} dtd\mathbb{P} \\
&= \int_0^\infty \int_A 1_{\{t<X\}} d\mathbb{P}dt \\
&= \int_0^\infty \int_A \mathbb{E}\{1_{\{t<X\}}|\mathcal{G}\} d\mathbb{P}dt \\
&= \int_A \int_0^\infty \mathbb{E}\{1_{\{t<X\}}|\mathcal{G}\} dtd\mathbb{P},
\end{aligned}$$

(0.4) follows.

(7) (10 points) Let $\{X_n, n \ge 0\}$ be a time homogeneous Markov process with state space $E = \{0, 1, 2, 3, \cdots\}$. Define

$$T_k = \inf\{n \ge 1 : X_n = k\}.$$

Prove that

$$\mathbb{E}^a\{\sum_{n=0}^{T_k-1} 1_{\{X_n=j\}}\} = \mathbb{E}^a\{\sum_{n=0}^{\infty} 1_{\{X_n=j, n\le T_k-1\}}\},$$

where $a \in E$ is the initial state.
Proof. By Fubini's theorem, we have

$$\begin{aligned}
\mathbb{E}^a\{\sum_{n=0}^{T_k-1} 1_{\{X_n=j\}}\} &= \sum_{l=0}^{\infty} \mathbb{E}^a\big[(\sum_{n=0}^{T_k-1} 1_{\{X_n=j\}})1_{\{T_k-1=l\}}\big] \\
&= \sum_{l=0}^{\infty} \mathbb{E}^a\big[\sum_{n=0}^{l} 1_{\{X_n=j\}\cap\{T_k-1=l\}}\big] \\
&= \mathbb{E}^a \sum_{n=0}^{\infty}\sum_{l=n}^{\infty} 1_{\{X_n=j\}\cap\{T_k-1=l\}} \\
&= \mathbb{E}^a \sum_{n=0}^{\infty} 1_{\{X_n=j, T_k-1\geq n\}}.
\end{aligned}$$

(8) (10 points) (Basic Lemma) Let (S,ρ) be a metric space and $\{\xi_n, n \in \mathbb{N}\}$ be a S-valued random variables. Then, $\xi_n \to \xi$ in probability if and only if every subsequence $\{\xi_{n'}, n' \in \mathbb{N}' \subset \mathbb{N}\}$ has a further subsequence $\{\xi_{n''}, n'' \in \mathbb{N}'' \subset \mathbb{N}'\}$ such that $\xi_{n''} \to \xi$ a.s. as n'' along $\mathbb{N}''$ goes to infinity.
Proof. "$\Rightarrow$" Assume that $\xi_n \to \xi$ in probability along $\mathbb{N}$. For any subset $\mathbb{N}' \subset \mathbb{N}$, we may choose a further subsequence $\mathbb{N}'' \subset \mathbb{N}'$ such that $\sum_{n''\in\mathbb{N}''} \mathbb{E}[\rho(\xi_{n''},\xi)\wedge 1] < \infty$ by dominated convergence theorem since $\rho(\xi_{n''},\xi)\wedge 1$ is bounded and $\xi_n \to \xi$ in probability. By Fubini's theorem, we have

$$\mathbb{E}\sum_{n''\in\mathbb{N}''}\{\rho(\xi_{n''},\xi)\wedge 1\} = \sum_{n''\in\mathbb{N}''}\mathbb{E}\{\rho(\xi_{n''},\xi)\wedge 1\}.$$

The series on the left-hand side then converges a.s. which implies $\xi_{n''} \to \xi$ a.s.
"$\Leftarrow$" If $\xi_n \not\to \xi$ in probability, then there exists some $\epsilon > 0$ such that $\mathbb{E}[\rho(\xi_{n'},\xi)\wedge 1] > \epsilon$ along a subsequence $\mathbb{N}' \subset \mathbb{N}$. By hypothesis, there exists a further subsequence $\mathbb{N}'' \subset \mathbb{N}'$ such that $\xi_{n''} \to \xi$ a.s. along $\mathbb{N}''$, and by dominated convergence theorem we get $\mathbb{E}\{\rho(\xi_{n''},\xi)\wedge 1\} \to 0$ along $\mathbb{N}''$, a contradiction.

(9) (10 points) For any metric spaces S and T, let $\xi, \xi_1, \xi_2, \cdots$ be random elements in S with $\xi_n \to \xi$ in probability, and let the mapping $f : S \to T$ be measurable and *a.s.* continuous at ξ. Prove that $f(\xi_n) \to f(\xi)$ in probability.
Solution. Since $\xi_n \to \xi$ in probability, by question (8) for any subsequence $\mathbb{N}' \subset \mathbb{N}$ there exists a further subsequence $\mathbb{N}'' \subset \mathbb{N}'$ such that $\xi_{n''} \to \xi$ a.s. along $\mathbb{N}''$. Thus, by the continuity of f we have $f(\xi_{n''}) \to f(\xi)$ a.s. along $\mathbb{N}''$. Hence $f(\xi_n) \to f(\xi)$ in probability by question (8).

(10) (10 points) Let $\xi, \xi_1, \xi_2, \cdots$ be random elements in a metric space (S,ρ). Prove that $\xi_n \to \xi$ in probability implies $\xi_n \to \xi$ in distribution, and the two conditions are equivalent when ξ is *a.s.* constant.

Proof. Let $C_b(S)$ denote all the bounded continuous functions on S. Assume $\xi_n \to \xi$ in probability. For any $f \in C_b(S)$ we need to show that $\mathbb{E}f(\xi_n) \to \mathbb{E}f(\xi)$. If the convergence fails, we may choose some subsequence $\mathbb{N}' \subset \mathbb{N}$ such that

$$\inf_{n' \in \mathbb{N}'} |\mathbb{E}f(\xi_{n'}) - \mathbb{E}f(\xi)| > 0.$$

By question (8), there exists a further subsequence $\mathbb{N}'' \subset \mathbb{N}'$ such that $\xi_{n''} \to \xi$ a.s. along $\mathbb{N}''$. By the continuity of f and dominated convergence theorem we get that $\mathbb{E}f(\xi_{n''}) \to \mathbb{E}f(\xi)$ along $\mathbb{N}''$, a contradiction. This proves that $\mathbb{E}f(\xi_n) \to \mathbb{E}f(\xi)$ for any $f \in C_b(S)$ or $\xi_n \to \xi$ in distribution.

Conversely, if $\xi_n \to s$, where $s \in S$ is a fixed element, since $\rho(x, s) \wedge k$ is a bounded and continuous function of x, we get

$$\mathbb{E}[\rho(\xi_n, s) \wedge k] \to \mathbb{E}[\rho(s, s) \wedge k] = 0$$

by definition. By question (8), suppose that $\xi_n \nrightarrow s$ in probability, then there exist a subsequence $\mathbb{N}' \subset \mathbb{N}$, a $k \geq 1$ and an $\epsilon > 0$ such that $\inf_{n'}[\rho(\xi_{n'}, s) \wedge k] \geq \epsilon$ a.s. However,

$$1 = \mathbb{P}(\inf_{n'}[\rho(\xi_{n'}, s) \wedge k] \geq \epsilon) \leq \mathbb{P}([\rho(\xi_{n'}, s) \wedge k] \geq \epsilon) \leq \frac{1}{\epsilon}\mathbb{E}[\rho(\xi_{n'}, s) \wedge k] \to 0$$

as $n' \to \infty$. This contradiction shows that $\xi_{n'} \to \xi$ in probability.

(11) (10 points) Let $\{B_t\}$ be a Brownian motion. Show that almost surely,

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log t}} \leq 1.$$

Proof. Since $B_n/\sqrt{n}$ is a standard Gaussian,

$$\mathbb{P}\{B_n > (1+\varepsilon)\sqrt{2n \log n}\} \leq c\frac{e^{-(1+\varepsilon)\log n}}{\log n} \leq \frac{c}{n^{1+\varepsilon}}.$$

By Borell-Cantelli, $B_n \leq (1+\varepsilon)\sqrt{2n \log n}$ eventually for all ε.

Note that $Y_n = \sup_{s \leq 1}(B_s - B_n)$ are i.i.d. random variables each with the same distribution as $|B_1|$.

$$\mathbb{P}\{Y_n > \sqrt{2n}\} \leq ce^{-n},$$

so by Borel-Cantelli, $Y_n \leq \sqrt{2n}$ eventually. Thus

$$B_s = B_n + B_s - B_n \leq \sqrt{2s \log s}(1 + o(1))(1+\varepsilon)$$

eventually.

(12) (10 points) Let $\{B_t\}$ be Brownian motion in $\mathbb{R}^2$. Let

$$\tau_a = \inf\{t > 0 : |B_t| = a\}.$$

Suppose that $a < |x| < b$. Show that

$$\mathbb{P}_x \{\tau_b < \tau_a\} = \frac{\log(|x|^2) - \log(|a|^2)}{\log(|b|^2) - \log(|a|^2)}.$$

Proof. Let $h(x_1, x_2) = \log(x_1^2 + x_2^2)$. Note that

$$\frac{\partial^2 h}{\partial x_1^2} = 2x_2^2 - 2x_1^2$$
$$\frac{\partial^2 h}{\partial x_2^2} = 2x_1^2 - 2x_2^2$$

Thus $\Delta h = 0$ and $M_t = h(B_t)$ is a martingale by Itô's Formula. Let $\tau = \tau_a \wedge \tau_b$. Applying the optional stopping formula,

$$\log(|x|^2) = \mathbb{E}(h(B_0)) = \mathbb{E}(h(B_\tau)) = h(b)\mathbb{P}_x\{\tau_b < \tau_a\} + h(a)\left[1 - \mathbb{P}_x\{\tau_b < \tau_a\}\right].$$

Solving for $\mathbb{P}_x\{\tau_b < \tau_a\}$ completes the solution.