Qualifying Examination

Theory of Probability

Fall 2005

NAME:______________________________

Instructions:

(1) This is a **close** book and **close notes** exam.

(2) This examination consists a total of **SEVEN** questions and comprises **three** printed pages.

(3) Answer **all** questions in **3 hours**. Solve problems step by step and show all your work.

Throughout this paper, all random variables are real-valued unless otherwise specified.

1. (15 points)

   (a) Prove that
   $$\forall\, x \in R^1,\ e^x + e^{-x} \le 2e^{x^2/2}$$

   (b) Let X be a random variable with $P(X=1) = P(X=-1) = 1/2$. Prove that $E(e^{tX}) \le e^{t^2/2}$.

   (c) Let $X_1, \cdots, X_n$ be i.i.d random variables with $P(X_i = 1) = P(X_i = -1) = 1/2$. Prove that
   $$\forall\, x > 0,\ P(S_n \ge x\sqrt{n}) \le e^{-x^2/2}$$
   where $S_n = \sum_{i=1}^n X_i$. (Hint: apply the Chebyshev inequality to e^{tS_n})

2. (12 points)

   (a) Let $X_1, X_2, \cdots$ be independent random variables with zero means and bounded fourth moments. Prove that
   $$\frac{1}{n^{4/5}} S_n \to 0 \quad a.s.$$
   where $S_n = \sum_{i=1}^n X_i$

   (b) Let $U_1, U_2, \cdots$ be independent and uniformly distributed on $[0, \pi]$ and let
   $$I_n = \frac{\pi^2}{4n} \sum_{i=1}^n \sin(U_i).$$
   Prove that (i) $I_n \to \frac{\pi}{2}$ a.s., and (ii) $E\{\exp(\sin(I_n))\} \to e$.

3. (13 points)

   (a) Let $\{S_i, 1 \le i \le n\}$ be a sequence of random variables. Assume that there exists $C > 0$ such that
   $$E|S_j - S_i|^2 \le C(j-i)$$
   for $0 \le i < j \le n$ (where $S_0 = 0$). Prove that
   $$E \max_{1 \le i \le n} S_i^2 \le 4Cn(\log(2n))^2.$$

   (b) Let $X_1, X_2, \cdots$ be independent random variables with $P(X_n = n) = P(X_n = -n) = 1/2$ for $n \ge 1$. Prove that
   $$\frac{1}{n^{3/2}\log^2 n} \sum_{i=1}^n X_i \to 0 \quad a.s.$$

4. (15 points) Let $h(w)$ be a bounded continuous function satisfying $|h| \leq 1$ and $|h'| \leq 1$, and Z be the standard normal random variable. Let f be the solution to the following Stein equality

$$f'(w) - wf(w) = h(w) - Eh(Z).$$

It is known that $|f''| \leq 4$. Using the Stein method to prove that if $X_1, X_2, \cdots X_n$ are independent random variables with $EX_i = 0$ and $\sum_{i=1}^n EX_i^2 = 1$ and $W = \sum_{i=1}^n X_i$, then

$$|Eh(W) - Eh(Z)| \leq 6 \sum_{i=1}^n E|X_i|^3.$$

5. (15 points) Let X_t be an integrable $\mathcal{F}_t$-adapted process for $t \geq 0$. Show that X_t is a martingale if and only if $\mathbb{E}X_\tau = \mathbb{E}X_\sigma$ for every pair of bounded stopping times with $\sigma \leq \tau$.

6. (15 points) Assume that $X_n \sim N(0,1)$ are independent. Define $Y_n = 1_{(X_n \geq \sqrt{\ln n^c})}$. Find a set $D \subset \mathbb{R}$ such that when $c \in D$, $\lim_{n\to\infty} Y_n = 0 \quad a.s.$.

7. (15 points) If $f(t)$ is a characteristic function, show that $\phi(t) = e^{f(t)-1}$ is also a characteristic function.

Qualifying Examination Key

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Throughout this paper, all random variables are real-valued unless otherwise specified.

1. (15 points)

   (a) Prove that

   $$\forall\ x \in R^1,\ e^x + e^{-x} \le 2e^{x^2/2}$$

   solution It is easy to see that

   $$\begin{aligned} & 2e^{x^2/2} - (e^x + e^{-x}) \\ = & \ 2\sum_{k=0}^{\infty}[\frac{(x^2/2)^k}{k!} - \frac{x^{2k}}{(2k)!}] \\ = & \ 2\sum_{k=2}^{\infty}[\frac{1}{2^k k!} - \frac{1}{(2k)!}]x^{2k} \\ \le & \ 0 \end{aligned}$$

   by the fact that $2^k k! < (2k)!$ for $k \ge 2$.

   (b) Let X be a random variable with $P(X = 1) = P(X = -1) = 1/2$. Prove that $E(e^{tX}) \le e^{t^2/2}$.

   solution For any t, we have

   $$E(e^{tX}) = (1/2)(e^t + e^{-t}) \le e^{t^2/2}$$

   by (a).

   (c) Let $X_1, \cdots, X_n$ be i.i.d random variables with $P(X_i = 1) = P(X_i = -1) = 1/2$. Prove that

   $$\forall\ x > 0,\ P(S_n \ge x\sqrt{n}) \le e^{-x^2/2}$$

   where $S_n = \sum_{i=1}^n X_i$. (Hint: apply the Chebyshev inequality to e^{tS_n})

   solution For any $x > 0$ and $t > 0$, we have

   $$\begin{aligned} P(S_n \ge x\sqrt{n}) &= P(e^{tS_n} \ge e^{tx\sqrt{n}}) \\ &\le e^{-tx\sqrt{n}} Ee^{tS_n} \\ &= e^{-tx\sqrt{n}}(Ee^{tX_1})^n \\ &\le e^{-tx\sqrt{n}} e^{t^2 n/2} \qquad [\text{by}(b)] \\ &= e^{-tx\sqrt{n}+t^2 n/2}, \end{aligned}$$

   which yields the desired bound $e^{-x^2/2}$ with $t = x/\sqrt{n}$.

2. (12 points)

(a) Let $X_1, X_2, \cdots$ be independent random variables with zero means and bounded fourth moments. Prove that

$$\frac{1}{n^{4/5}} S_n \to 0 \quad a.s.$$

where $S_n = \sum_{i=1}^{n} X_i$

solution Let $K = \max_{i \geq 1} E|X_i|^4$. By the Rosenthal inequality, there is a constant C depending only on K such that

$$E|S_n|^4 \leq Cn^2$$

Thus, for any $p > 0$, by the Chebyshev inequality

$$P(|S_n| > pn^{4/5}) \quad \leq \quad (pn^{4/5})^{-4} ES_n^4 \leq (pn^{4/5})^{-4} Cn^2 = (p)^{-4} Cn^{-6/5}$$

Therefore

$$\sum_{n=1}^{\infty} P(|S_n| > pn^{4/5}) < \infty$$

and hence

$$\frac{1}{n^{4/5}} S_n \to 0 \; a.s.$$

by the Borel-Cantelli lemma.

(b) Let $U_1, U_2, \cdots$ be independent and uniformly distributed on $[0, \pi]$ and let

$$I_n = \frac{\pi^2}{4n} \sum_{i=1}^{n} \sin(U_i).$$

Prove that (i) $I_n \to \frac{\pi}{2}$ a.s., and (ii) $E\{\exp(\sin(I_n))\} \to e$.

solution (i) It is easy to see that

$$E \sin(U_1) = \frac{1}{\pi} \int_0^{\pi} \sin x dx = 2/\pi.$$

Since $\sin(U_i)$ are iid bounded random variables, by the strong law of large numbers

$$I_n \to \frac{\pi^2}{4} E \sin(U_1) = \pi/2 \quad a.s.$$

(ii) Because $I_n \to \pi/2$ a.s, we have $\exp(\sin(I_n)) \to e$ a.s. Also note that $\exp(\sin(x))$ is bounded, by the Lebesque dominated convergence theorem

$$E \exp(\sin(I_n)) \to e$$

3. (a) (13 points) Let $\{S_i, 1 \leq i \leq n\}$ be a sequence of random variables. Assume that there exists $C > 0$ such that

$$E|S_j - S_i|^2 \leq C(j - i)$$

for $0 \le i < j \le n$ (where $S_0 = 0$). Prove that

$$E \max_{1 \le i \le n} S_i^2 \le 4Cn(\log(2n))^2.$$

solution We prove it by induction. The result is trivial for $n \le 4$. So, assume $n > 4$ and it holds for any k less than n. We now prove that it remains valid for n. Let $k = [n/2]$. Observe that

$$\max_{1 \le i \le n} |S_i| = \max(\max_{1 \le i \le k} |S_i|, \max_{k < i \le n} |S_i - S_k + S_k|)$$

and

$$\begin{aligned} \max_{1 \le i \le n} |S_i|^2 &\le \max_{1 \le i \le k} S_i^2 + \max_{k < i \le n} (|S_i - S_k| + |S_k|)^2 \\ &\le \max_{1 \le i \le k} S_i^2 + \max_{k < i \le n} |S_i - S_k|^2 + 2|S_k| \max_{k < i \le n} |S_i - S_k| + S_k^2. \end{aligned}$$

Thus

$$\begin{aligned} E \max_{1 \le i \le n} |S_i|^2 \le\ & E \max_{1 \le i \le k} S_i^2 + E \max_{k < i \le n} |S_i - S_k|^2 \\ & +2E|S_k| \max_{k < i \le n} |S_i - S_k| + ES_k^2 \\ & \le E \max_{1 \le i \le k} S_i^2 + E \max_{k < i \le n} |S_i - S_k|^2 \\ & +2(E|S_k|^2)^{1/2} (E \max_{k < i \le n} |S_i - S_k|^2)^{1/2} + ES_k^2 \\ & \le 4Ck \log^2(2k) + 4C(n-k)\log^2(2(n-k)) \\ & +2(Ck)^{1/2}(4C(n-k)\log^2(2(n-k)))^{1/2} + Ck \\ & \text{[by induction hypothesis]} \\ & \le 4Ck(\log(2n) - \log(1.5))^2 + 4C(n-k)(\log(2n) - \log(1.5))^2 \\ & +2Cn(\log(2n) - \log 1.5) + Cn/2 \\ & \le 4Cn\Big((\log(2n) - \log(1.5))^2 + 0.5(\log(2n) - \log(1.5)) + 1/8\Big) \\ & \le 4Cn(\log(2n) - \log 1.5 + 2^{-3/2})^2 \\ & \le 4Cn \log^2(2n) \end{aligned}$$

as desired.

(b) Let $X_1, X_2, \cdots$ be independent random variables with $P(X_n = n) = P(X_n = -n) = 1/2$ for $n \ge 1$. Prove that

$$\frac{1}{n^{3/2} \log^2 n} \sum_{i=1}^{n} X_i \to 0 \quad a.s.$$

solution:

We use the subsequence method and the result in part (a). Note that for any fixed n and $1 \le i < j \le n$.

$$E(S_j - S_i)^2 = \sum_{k=i+1}^{j} k^2 \le n^2(j-i).$$

So by (a)

$$E \max_{1\le i\le n} |S_i|^2 \le 4n^3 \log^2(2n)$$

and

$$\begin{aligned} & P(\max_{1\le i\le 2^k} |S_i| > 2^{3k/2}k^{1.6}) \\ \le & \frac{4 2^{3k} \log^2(2^{k+1})}{2^{3k}k^{3.2}} \\ \le & C/k^{1.2} \end{aligned}$$

Therefore

$$\sum_{k=1}^{\infty} P(\max_{1\le i\le 2^k} |S_i| > 2^{3k/2}k^{1.6}) < \infty$$

Now by the Borel-Cantelli lemma and the subsequence method, we have

$$\frac{1}{n^{1.5}\log^2(n)}|S_n| \to 0 \quad a.s.$$

4. (15 points) Let $h(w)$ be a bounded continuous function satisfying $|h| \le 1$ and $|h'| \le 1$, and Z be the standard normal random variable. Let f be the solution to the following Stein equality

$$f'(w) - wf(w) = h(w) - Eh(Z).$$

It is known that $|f''| \le 4$. Using the Stein method to prove that if $X_1, X_2, \cdots X_n$ are independent random variables with $EX_i = 0$ and $\sum_{i=1}^n EX_i^2 = 1$ and $W = \sum_{i=1}^n X_i$, then

$$|Eh(W) - Eh(Z)| \le 6\sum_{i=1}^{n} E|X_i|^3.$$

solution: Write $W_{(i)} = W - X_i$. Then $W_{(i)}$ and X_i are independent. Note that

$$\begin{aligned} EWf(W) &= \sum_{i=1}^{n} EX_i f(W) \\ &= \sum_{i=1}^{n} E(X_i(f(W) - f(W_{(i)}))) \\ &= \sum_{i=1}^{n} EX_i \int_0^{X_i} f'(W_{(i)} + t)dt \\ &= \sum_{i=1}^{n} E\int_{-\infty}^{\infty} f'(W_{(i)} + t)M_i(t)dt \end{aligned}$$

where $M_i(t) = EX_i(I(0 < t \le X_i) - I(X_i < t \le 0))$. Hence by the Stein equality

$$\begin{aligned} |Eh(W) - Eh(Z)| &= |Ef'(W) - EWf(W)| \\ &= |\sum_{i=1}^{n} \int_{-\infty}^{\infty} E(f'(W) - f'(W_{(i)} + t))M_i(t)dt| \\ &\le \sum_{i=1}^{n} \int_{-\infty}^{\infty} 4E(|X_i| + |t|)M_i(t)dt \\ &= 4\sum_{i=1}^{n}(E|X_i|EX_i^2 + E|X_i|^3/2) \\ &\le 6\sum_{i=1}^{n} E|X_i|^3 \end{aligned}$$

5. (15 points) Let X_t be an integrable $\mathcal{F}_t$-adapted process for $t \ge 0$. Show that X_t is a martingale if and only if $\mathbb{E}X_\tau = \mathbb{E}X_\sigma$ for every pair of bounded stopping times with $\sigma \le \tau$.

Solution. " $\Rightarrow$ ". Define $H_n \triangleq 1_{(\tau \ge n)}$, $Y_0 = X_0$, and $Y_n = X_0 + \sum_{k=1}^{n} H_k(X_k - X_{k-1})$. Then, Y_n is a martingale by martingale transformation. By Fubini's theorem, we have

$$\begin{aligned} Y_n &= X_0 + \sum_{k=1}^{n} 1_{(k \le \tau)}(X_k - X_{k-1}) \\ &= X_0 + \sum_{k=1}^{n}\sum_{i=k}^{\infty} 1_{(\tau = i)}(X_k - X_{k-1}) \\ &= X_0 + \sum_{i=1}^{n}\sum_{k=1}^{i} 1_{(\tau = i)}(X_k - X_{k-1}) \\ &+ \sum_{i=n+1}^{\infty}\sum_{k=1}^{n} 1_{(\tau = i)}(X_k - X_{k-1}) \\ &= X_0 + \sum_{i=1}^{n} 1_{(\tau = i)}(X_i - X_0) \\ &+ \sum_{i=n+1}^{\infty} 1_{(\tau = i)}(X_n - X_0) \\ &= X_0 + X_{n\wedge\tau} - X_0 = X_{n\wedge\tau} \end{aligned}$$

Let $\tilde{H}_n \triangleq 1_{(\sigma < n \le \tau)} = 1_{(\tau \ge n)} - 1_{(\sigma \ge n)}$. Then, $\tilde{H}_n$ is predictable.

$$\tilde{Y}_n = X_0 + X_{n\wedge\tau} - X_{n\wedge\sigma}$$

is a martingale. If $\tau \le M \in \mathbb{N}$, then

$$\mathbb{E}\tilde{Y}_M = \mathbb{E}\tilde{Y}_0 = \mathbb{E}X_0.$$

Since

$$\tilde{Y}_M - X_0 = X_\tau - X_\sigma,$$

we have

$$\mathbb{E}X_\tau = \mathbb{E}X_\sigma.$$

" $\Leftarrow$ ". Suppose that $s < t$ and $B \in \mathcal{F}_s$. Let $\sigma \equiv s$ and

$$\tau = \begin{cases} s & \text{if } \omega \in B^c \\ t & \text{if } \omega \in B \end{cases}.$$

Then,

$$\mathbb{E}(X_\sigma) = \mathbb{E}(X_s) = \mathbb{E}(X_\tau) = \mathbb{E}(X_s 1_{B^c}) + \mathbb{E}(X_t 1_B).$$

This gives

$$\mathbb{E}(X_s 1_B) = \mathbb{E}(X_t 1_B), \quad B \in \mathcal{F}_s.$$

Thus, $\{X_t\}$ is a $\mathcal{F}_t$-martingale.

6. (15 points) Assume that $X_n \sim N(0,1)$ are independent. Define $Y_n = 1_{(X_n \geq \sqrt{\ln n^c})}$. Find a set $D \subset \mathbb{R}$ such that when $c \in D$, $\lim_{n\to\infty} Y_n = 0 \quad a.s.$.

Solution: Since

$$\mathbb{P}(\lim_{n\to\infty} Y_n = 0) = 1 \Leftrightarrow \mathbb{P}(\limsup_{n\to\infty}\{X_n \geq \sqrt{\ln n^c}\}) = 0,$$

by Borel-Cantelli's lemma, the following is a sufficient condition for above result.

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n \geq \sqrt{\ln n^c}) < \infty \Leftrightarrow \sum_{n=1}^{\infty} \int_{\sqrt{\ln n^c}}^{\infty} e^{-x^2/2} dx < \infty.$$

Since

$$\int_{\sqrt{\ln n^c}}^{\infty} e^{-x^2/2} dx \int_{\sqrt{\ln n^c}}^{\infty} e^{-y^2/2} dy = \int_{\sqrt{\ln n^c}}^{\infty} \int_{\sqrt{\ln n^c}}^{\infty} e^{-(x^2+y^2)/2} dx dy$$
$$= \int_0^{\pi/2} d\theta \int_{\sqrt{\ln n^{2c}}}^{\infty} e^{-r^2/2} r dr = \frac{\pi}{2} \frac{1}{n^c},$$

and

$$\sum_{n=1}^{\infty} \sqrt{\frac{\pi}{2}} \frac{1}{n^{c/2}} < \infty \qquad \text{for} \qquad c > 2.$$

On the other hand,

$$\sum_{n=1}^{\infty} \sqrt{\frac{\pi}{2}} \frac{1}{n^{c/2}} = \infty \qquad \text{for} \qquad c \leq 2.$$

Since $\{X_i\}$ are independent, by Borel-Cantelli's lemma, we have

$$\mathbb{P}(\limsup_{n\to\infty}\{X_n \geq \sqrt{\ln n^c}\}) = 1, \qquad c \leq 2.$$

Thus, we have $D = (2, \infty)$.

7. (15 points) If $f(t)$ is a characteristic function, show that $\phi(t) = e^{f(t)-1}$ is also a characteristic function.

Proof: (i) If $a_i \geq 0$, $\sum_{i=1}^{n} a_i = 1$, and if $f_i(t), i = 1, \cdots n$ are characteristic functions, then $\sum_{i=1}^{n} a_i f_i(t)$ is the characteristic function of distribution function $\sum_{i=1}^{n} a_i F_i(x)$, where $F_i(x)$ is the distribution function corresponding to characteristic function $f_i(t)$.
(ii) If $a_i \geq 0$, $\sum_{i=1}^{\infty} a_i = A < \infty$ and if $f_i(t), i = 1, \cdots$ are characteristic functions, let $A_n = \sum_{i=1}^{n} a_i$, then

$$\phi_n(t) \triangleq \sum_{i=1}^{n} \frac{a_i}{A_n} f_i(t) \qquad \text{for} \qquad n \geq 1$$

are characteristic functions. Since $|f_i(t)| \leq 1$ and $f_i(t)$ is continuous, by the convergence test of series with function terms, $\phi_n(t)$ converges uniformly to a continuous function $\phi(t)$. By the continuity theorem, $\phi(t)$ is a characteristic function.
(iii) If $f_i(t), i = 1, \cdots$ are characteristic functions, $\prod_{i=1}^{n} f_i(t)$ is obvious a characteristic function of sum of n independent rv's $\xi_i, i = 1, \cdots n$, where ξ_i has characteristic function $f_i(t)$.
(iv) Let $g(x) = e^x$. If $f(t)$ is a characteristic function, by above (i) (ii), and (iii),

$$e^{f(t)-1} = \frac{1}{g(1)} g(f(t)) = \frac{1}{g(1)} \sum_{k=0}^{\infty} \frac{(f(t))^k}{k!}$$

is a characteristic function.