Qualifying Examination

Theory of Probability

September 2004

NAME:____________________________

Instructions:

(1) This is a **close** book and **close notes** exam.

(2) This examination consists a total of **six** questions and comprises **two** printed pages.

(3) Answer **all** questions in **3 hours**. Solve problems step by step and show all your work.

Throughout this paper, all random variables are real-valued unless otherwise specified.

1. (a) (9 points) Let $A_1, A_2, \ldots$ be a sequence of events on a probability space $(\Omega, \mathcal{F}, P)$.

   (i) Prove that if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n \ i.o.) = 0$.

   (ii) Prove that if $\sum_{i=1}^{n} P(A_i) \to \infty$ and $\frac{\sum_{i=1}^{n}\sum_{j=1}^{n} P(A_i A_j)}{(\sum_{i=1}^{n} P(A_i))^2} \to 1$ as $n \to \infty$, then $P(A_n \ i.o.) = 1$.

   (b) (5 points) Let $X_1, X_2, \cdots$ be independent distributed random variables with a common probability density function

$$f(x) = \frac{c_0}{(1+x^2)\ln(3+x^2)} \quad \text{for } -\infty < x < \infty,$$

   where $c_0 > 0$ is a constant. Prove that

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to 0 \text{ in probability.}$$

2. (a) (10 points) Let $X_1, X_2, \cdots$ be independent and identically distributed random variables and let $S_n = \sum_{i=1}^{n} X_i$. Prove that

$$\frac{S_n}{n} \to \mu \ a.s.$$

   for some finite constant μ if and only if $E(|X_1|) < \infty$.

   (b) (8 points) Let $X_1, X_2, \cdots$ be independent and identically distributed random variables with $E(X_1) = \mu$ and $\mathrm{Var}(X_1) = \sigma^2 < \infty$. Put

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \ s_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}, \ T_n = n^{1/2}(\bar{X} - \mu)/s_n.$$

   Prove

   (i) $s_n/\sigma \to 1$ a.s. as $n \to \infty$;

   (ii) $T_n \xrightarrow{d.} N(0,1)$.

3. (a) (5 points) Assume that random vectors (X, Y) and (X, Z) have the same joint distribution. Prove that if $g : R^1 \to R^1$ is a measurable function with $E|g(Y)| < \infty$, then $E(g(Y) \mid X) = E(g(Z) \mid X)$.

   (b) (4 points) Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed random variables with $E(|X_i|) < \infty$. Let $S_n = \sum_{i=1}^{n} X_i$. Compute $E(X_1 | S_n)$.

   (c) (4 points) Let $\{T_n, \mathcal{F}_n, n \geq 0\}$ be a submartingale and N be a stoping time. Prove that $E(T_0) \leq E(T_{\min(n,N)}) \leq E(T_n)$ for $n \geq 1$.

(d) (7 points) (Let $X_1, X_2, \cdots$ be independent random variables with $P(X_i = 1) = p$ and $P(X_i = -1) = 1-p$, where $1/2 < p < 1$. Let $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$ for $n = 1, 2, \cdots$, $\mathcal{F}_n = \sigma(X_i i \le n)$ for ≥ 1 and $\mathcal{F}$ be the trivial σ-field.

(i) Let $\phi(x) = ((1-p)/p)^x$. Prove that $\{\phi(S_n), \mathcal{F}_n, n \ge 0\}$ is a martingale;

(ii) Let $T_x = \inf\{n \ge 1 : S_n = x\}$. Prove that for all positive integer k

$$P(T_{-k} < T_k) = \frac{1}{1 + \phi(-k)}.$$

4. (14 points) Let $(X_n)_{n\ge 0}$ be a Markov chain on $\{0, 1, 2\}$ with the transition matrix

$$p = \begin{pmatrix} 0.5 & 0.25 & 0.25 \\ 0.1 & 0.55 & 0.35 \\ 0 & 0.5 & 0.5 \end{pmatrix}.$$

(i) Which sets are closed? which sets are irreducible? which states are recurrent?

(ii) Find the stationary distribution π of $(X_n)_{n\ge 0}$;

(iii) Let $T_x = \inf\{n \ge 1 : X_n = x\}$. Find $E_x(T_x)$ for $x = 0, 1, 2$.

(iv) Find the initial distribution of X_0 so that $(X_n,\, n \ge 0)$ is a stationary Markov chain.

5. (14 points) Let $\{B(t), t \ge 0\}$ be a Brownian motion and $\mathcal{F}_t = \sigma(B(s), 0 \le s \le t)$.

(i) Let τ be a finite stoping time with respect to $\{\mathcal{F}_t, t \ge 0\}$ and let $X(t) = B(t+\tau) - B(\tau)$. Prove that $\{X(t), t \ge 0\}$ is a Brownian motion and is independent of τ.

(ii) Prove that $\{B(t), t \ge 0\}$ and $\{B^2(t) - t, t \ge 0\}$ are martingales.

(iii) Let $a > 0$ and $\tau = \inf\{t \ge 0 : B(t) \notin (-a, a)\}$. Prove that $P(B(\tau) = a) = 1/2$ and $E(\tau) = a^2$.

6. (18 points) Let $X_1, X_2, \cdots$ be independent and identically distributed standard normal random variables and let $S_n = \sum_{i=1}^n X_i$.

(i) Prove that for all $x > 0$

$$P(\max_{1\le i\le n} S_i \ge x\sqrt{n}) \le 2P(S_n \ge x\sqrt{n})$$

and

$$P(\max_{1\le i\le n} |S_i| \ge x\sqrt{n}) \le \frac{2}{x} e^{-x^2/2}.$$

(ii) Prove that

$$\limsup_{n\to\infty} \frac{\max_{1\le i\le n} |S_i|}{(2n \ln\ln n)^{1/2}} = 1 \ a.s.$$

Solutions

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1. (a) (i) (**4 points**)

$$P(A_n i.o.) = \lim_{n\to\infty} P(\cup_{m=n}^{\infty} A_m) \le \lim_{n\to\infty} \sum_{m=n}^{\infty} P(A_m) = 0$$

(ii) (**5 points**) Observe that

$$\begin{aligned}
& P(\sum_{i=1}^{n} 1_{\{A_i\}} \le (1/2)\sum_{i=1}^{n} P(A_i)) \\
= \ & P(\sum_{i=1}^{n}(1_{\{A_i\}} - P(A_i)) \le -(1/2)\sum_{i=1}^{n} P(A_i)) \\
\le \ & \frac{4}{(\sum_{i=1}^{n} P(A_i))^2} \mathrm{Var}(\sum_{i=1}^{n} 1_{\{A_i\}}) \\
= \ & \frac{4}{(\sum_{i=1}^{n} P(A_i))^2} \sum_{i=1}^{n}\sum_{j=1}^{n}\{P(A_iA_j) - P(A_i)P(A_j)\} \\
\to \ & 0
\end{aligned}$$

as $n \to \infty$ by the assumption. Hence

$$P(\sum_{i=1}^{n} 1_{\{A_i\}} \ge (1/2)\sum_{i=1}^{n} P(A_i)) \to 1.$$

Since $\sum_{i=1}^{n} P(A_i) \to \infty$, the above yields $\sum_{i=1}^{n} 1_{\{A_i\}} \to \infty$ in probability and hence a.s. (because $\sum_{i=1}^{n} 1_{\{A_i\}}$ is non-decreasing). Therefore, $P(A_n i.o.) = 1$.

1. (b) (**5 points**) Let $\bar{X}_i = X_i 1_{\{|X_i| \le n\}}$ for $i = 1, 2, \cdots, n$ and $T_n = \sum_{i=1}^{n} \bar{X}_i$. Then for $\varepsilon > 0$ with $S_n = \sum_{i=1}^{n} X_i$

$$\begin{aligned}
P(|S_n| \ge \varepsilon n) & \le P(\max_{1\le i\le n} |X_i| > n) + P(|T_n| \ge \varepsilon n) \\
& \le nP(|X_1| > n) + (\varepsilon n)^{-2} E(T_n^2) \\
& = 2n \int_n^{\infty} f(x)dx + 2(\varepsilon n)^{-2} \int_0^n x^2 f(x)dx \\
& \le \frac{4c_0}{\ln(3+n^2)} + 2c_0(\varepsilon n)^{-2} n \\
& \to 0
\end{aligned}$$

as $n \to \infty$, as desired.

2.(a) *Sufficiency* (**7 points**). Assume that $E|X_1| < \infty$ and also assume $X_i \geq 0$. Let $\mu = EX_1$. Let $Y_i = X_i 1\{0 \leq X_i \leq i\}$ and $T_n = \sum_{i=1}^n Y_i$. Then $ET_n/n \to \mu$, $\sum_{i=1}^\infty P(X_i > i) \leq EX_1 < \infty$ and hence by the Borel-Cantelli lemma $P(X_n \neq Y_n i.o.) = 0$. It suffices to show that

$$\frac{T_n}{n} \to \mu \quad a.s.$$

Observe that for $\varepsilon > 0$

$$\begin{aligned} P(|T_n - ET_n| \geq \varepsilon n) &\leq (\varepsilon n)^{-2}\text{Var}(T_n) \\ &\leq \varepsilon^{-2}\frac{1}{n}EX_1^2 1_{\{X_1 \leq n\}}. \end{aligned}$$

Let $n_k = [\theta^k]$, where $\theta > 1$. Then

$$\begin{aligned} \sum_{k=1}^\infty P(|T_{n_k} - ET_{n_k}| \geq \varepsilon n_k) &\leq \varepsilon^{-2}\sum_{k=1}^\infty \frac{1}{n_k}EX_1^2 1_{\{X_1 \leq n_k\}} \\ &\leq \varepsilon^{-2}\sum_{k=1}^\infty \frac{1}{n_k}\sum_{j=1}^k EX_1^2 1_{\{n_{j-1} < X_1 \leq n_j\}} \\ &\leq c_1\varepsilon^{-2}\sum_{j=1}^\infty \frac{1}{n_j}EX_1^2 1_{\{n_{j-1} < X_1 \leq n_j\}} \\ &\leq c_2\varepsilon^{-2}EX_1 < \infty \end{aligned}$$

where c_1, c_2 are constant. Therefore by the Borel-Cantelli lemma

$$\frac{T_{n_k} - ET_{n_k}}{n_k} \to 0 \ a.s.$$

Thus

$$\frac{T_{n_k}}{n_k} \to \mu \quad a.s.$$

For general m, let $n_{k-1} \leq m < n_k$. Then

$$\frac{T_{n_{k-1}}}{n_k} \leq \frac{T_m}{m} \leq \frac{T_{n_k}}{n_{k-1}}$$

and hence

$$\frac{\mu}{\theta} \leq \liminf_{m\to\infty}\frac{T_m}{m} \leq \limsup_{m\to\infty}\frac{T_m}{m} \leq \theta\mu \ a.s.$$

This proves $T_n/n \to \mu$ a.s. by the arbitrariness of $\theta > 1$.

For the general case, write $X_i = X_i^+ - X_i^-$. Then, the result follows from the first case.

Necessity. (**3 points**) Assume that $S_n/n \to \mu$ a.s. Then

$$|X_n - \mu|/n \leq |(S_n - n\mu)|/n + |S_{n-1} - (n-1)\mu|/n \to 0 \ a.s.$$

By the Borel-Cantelli lemma, we have

$$\infty > \sum_{n=1}^\infty P(|X_n - \mu|/n \geq 1) = \sum_{n=1}^\infty P(|X_1 - \mu| \geq n).$$

Hence $E|X_1 - \mu| < \infty$, which implies $E|X_1| < \infty$.

2. (b) (i) (**3 points**) We have

$$s_n^2 = \frac{1}{n-1}(\sum_{i=1}^n X_i^2 - \frac{n}{n-1}\bar{X}^2) \to EX_1^2 - (EX_1)^2 = \sigma^2 \ a.s.$$

by the law of large numbers.

(ii) (**5 points**) Write

$$T_n = \frac{n^{1/2}(\bar{X}-\mu)}{\sigma}\frac{\sigma}{s_n}$$

By the central limit theorem

$$\frac{n^{1/2}(\bar{X}-\mu)}{\sigma} \xrightarrow{d.} N(0,1)$$

and by (i), $\sigma/s_n \to 1$ in probability, we have $T_n \xrightarrow{d.} N(0,1)$

3.(a) (**5 points**)

Consider 4 different cases:

Case 1. g is an indicator function, i.e., $g(x) = 1_{\{A\}}(x)$, where A is a Borel set. Then for any Borel set B

$$\begin{aligned} E(g(Y)1\{X \in B\}) &= E(1\{Y \in A\}1\{X \in B\}) = P(Y \in A, X \in B) \\ &= P(Z \in A, X \in B) = E(g(Z)1\{X \in B\}) \end{aligned}$$

Therefore $E(g(Y)|X) = E(g(Z)|X)$.

Case 2. g is simple function, i.e., $g = \sum_{i=1}^n c_i g_i$, where g_i are indicator functions. Then

$$E(g(Y)|X) = \sum_{i=1}^n c_i E(g_i(Y)|X) = \sum_{i=1}^n c_i E(g_i(Z)|X) = E(g(Z)|X).$$

Case 3. g is non-negative. Then there exists non-decreasing simple functions g_n such that $g_n \to g$. By the monotone convergence theorem,

$$E(g(Y)|X) = \lim_{n\to\infty} E(g_n(Y)|X) = \lim_{n\to\infty} E(g_n(Z)|X) = E(g(Z)|X)$$

Case 4. general case. Write $g = g^+ - g^-$. Then

$$E(g(Y)|X) = E(g^+(Y)|X) - E(g^-(Y)|X) = E(g^+(Z)|X) - E(g^-(Z)|X) = E(g(Z)|X)$$

3.(b) (**4 points**) Since $\{X_i, 1 \le i \le n)$ are i.i.d., (X_1, S_n) and (X_i, S_n) have the same joint distribution for each i. Hence $E(X_1|S_n) = E(X_i|S_n)$ and

$$S_n = E(S_n|S_n) = \sum_{i=1}^n E(X_i|S_n) = \sum_{i=1}^n E(X_1|S_n) = nE(X_1|S_n).$$

This shows that $E(X_1|S_n) = S_n/n$.

3.(c) **(4 points)**

We have

$$ET_{\min(n,N)} = ET_N 1\{N < n\} + ET_n 1\{N \geq n\} = \sum_{j=1}^{n-1} ET_j 1\{N = j\} + E(T_n 1\{N \geq n\})$$

Since $\{T_n, \mathcal{F}_n, n \geq 0\}$ is a submartingale, $E(T_j 1\{N = j\}) \leq E(T_n 1\{N = j\})$ for $j \leq n-1$. Hence

$$\sum_{j=1}^{n-1} ET_j 1\{N = j\} + E(T_n 1\{N \geq n\} \leq \sum_{j=1}^{n-1} E(T_n 1\{N = j\}) + E(T_n 1\{N \geq n\} = ET_n$$

which proves that $ET_{\min(n,N)} \leq ET_n$. Noting that $\{N \geq n\} \in \mathcal{F}_{n-1}$, we have

$$E(T_n 1\{N \geq n\}) \geq E(T_{n-1} 1\{N \geq n\})$$

and hence

$$ET_{\min(n,N)} \geq ET_N 1\{N < n\} + E(T_{n-1} 1\{N \geq n\}) = ET_{\min(n-1,N)} \geq \cdots \geq ET_{\min(0,N)} = ET_0.$$

3.(d) (i) **(3 points)** We have

$$E(\phi(S_n)|\mathcal{F}_{n-1}) = \phi(S_{n-1})E(\phi(X_n)) = \phi(S_{n-1})$$

(ii) **(4 points)**. Let $N = T_{-k} \wedge T_k$. Then N is a stopping time and

$$\begin{aligned} P(N > 2mk) &= P(|S_i| < k, i = 1, \cdots, 2mk) \\ &\leq P(|S_{2ik} - S_{2(i-1)k}| < 2k, i = 1, \cdots m) \\ &= P(|S_{2k}| < 2k)^m = \left(1 - (p^{2k} + (1-p)^{2k})\right)^m \end{aligned}$$

Hence $E(N) < \infty$. Note that $\phi(S_{n \wedge N})$ is bounded, we have

$$\phi(0) = E\phi(S_N = P(T_{-k} < T_k)\phi(-k) + P(T_k < T_{-k})\phi(k)$$

Using $P(T_{-k} < T_k) + P(T_k < T_{-k}) = 1$ and solving the above equation gives the desired result.

4. (i) **(2 points)**

$\{0,1,2\}$ is closed, irreducible, and all states are recurrent.

(ii) **(6 points)**

Let π be the stationary distribution. Then $\sum_{i=0}^{2} \pi(i)p(i,j) = \pi(j)$ for $j = 0,1,2$. Solving

$$\begin{aligned}
0.5\pi(0) + 0.1\pi(1) &= \pi(0),\\
0.25\pi(0) + .55\pi(1) + .5\pi(2) &= \pi(1),\\
.25\pi(0) + .35\pi(1) + .5\pi(2) &= \pi(2),\\
\pi(0) + \pi(1) + \pi(2) &= 1
\end{aligned}$$

gives $\pi(0) = .1, \pi(1) = .5, \pi(2) = .4$

(iii) **(3 points)** $E_x(T_x) = \frac{1}{\pi(x)} = 10$ for $x = 0$, $= 2$ for $x = 1$ and $= 2.5$ for $x = 2$.

(iv) **(3 points)** $P(X_0 = i) = \pi(i)$ for $i = 0,1,2$. That is, $P(X_0 = 0) = 0.1, P(X_0 = 1) = .5, P(X_0 = 2) = .4$

5. (i) **(6 points)**

Let $n \geq 4$ and define

$$\tau_n = \frac{k}{n} \text{ if } \frac{k-1}{n} \leq \tau < \frac{k}{n},\ k = 1,2,\cdots$$

Then τ_n are stopping times. For $t_i > 0$ and Borel sets A_i and B

$$\begin{aligned}
& P(B(t_i+\tau_n) - B(\tau_n) \in A_i, 1 \leq i \leq m, \tau \in B)\\
= & \sum_{k=1}^{\infty} P(B(t_i+\tau_n) - B(\tau_n) \in A_i, 1 \leq i \leq m, (k-1)/n \leq \tau < k/n, \tau \in B)\\
= & \sum_{k=1}^{\infty} P(B(t_i+k/n) - B(k/n) \in A_i, 1 \leq i \leq m, (k-1)/n \leq \tau < k/n, \tau \in B)\\
= & \sum_{k=1}^{\infty} P(B(t_i+k/n) - B(k/n) \in A_i, 1 \leq i \leq m) P((k-1)/n \leq \tau < k/n, \tau \in B)\\
= & \sum_{k=1}^{\infty} P(B(t_i) \in A_i, 1 \leq i \leq m) P((k-1)/n \leq \tau < k/n, \tau \in B)\\
= & P(B(t_i) \in A_i, 1 \leq i \leq m) P(\tau \in B).
\end{aligned}$$

Letting $n \to \infty$ yields

$$P(B(t_i+\tau) - B(\tau) \in A_i, 1 \leq i \leq m, \tau \in B) = P(B(t_i) \in A_i, 1 \leq i \leq m) P(\tau \in B).$$

This proves that $\{B(t+\tau) - B(\tau), t \geq 0\}$ is a Brownian motion and is independent of τ.

(ii) **(4 points)** For $0 < s < t$

$$E(B(t)|\mathcal{F}_s) = B(s) + E(B(t) - B(s)|\mathcal{F}_s) = B(s) + E(B(t) - B(s)) = B(s)$$

Hence $\{B(t), t \geq 0\}$ is a martingale.

Similarly, for $0 < s < t$

$$\begin{aligned} E(B^2(t) - t|\mathcal{F}_s) &= E((B(s) + B(t) - B(s))^2|\mathcal{F}_s) - t \\ &= B^2(s) + 2B(s)E(B(t) - B(s)) + E(B(t) - B(s))^2 - t = B^2(s) - s. \end{aligned}$$

This proves that $\{B^2(t), t \geq 0\}$ is also a martingale

(iii) **(4 points)**

Since $\limsup_{t\infty} B(t)/\sqrt{t} = \infty$ and $\liminf_{t\to\infty} B(t)/\sqrt{t} = -\infty$, $\tau < \infty$ $a.s.$, we have $EB(\tau \wedge t) = 0$ for any $t > 0$. Letting $t \to \infty$ and using the bounded convergence theorem gives $EB(\tau) = 0$, which combines $P(B(\tau) = a) + P(B(\tau) = -a) = 1$ yields $P(B(\tau) = a) = 1/2$. Since $B^2(t) - t$ is a martingale, we have

$$E(\tau) = EB^2(\tau) = a^2(1/2) + (-a)^2(1/2) = a^2.$$

6. (i) **(9 points)** Let $A_k = \{S_j < x\sqrt{n}, 1 \leq j < k, S_k \geq x\sqrt{n}\}$. Then A_k are disjoint and $\{\max_{1\leq k\leq n} S_k \geq x\sqrt{n}\} = \cup_{1\leq k\leq n} A_k$. We have

$$\begin{aligned} P(\max_{1\leq k\leq n} S_k \geq x\sqrt{n}) &\leq P(S_n \geq x\sqrt{n}) + \sum_{1\leq k\leq n} P(A_k, S_n < x\sqrt{n}) \\ &\leq P(S_n \geq x\sqrt{n}) + \sum_{1\leq k\leq n} P(A_k, S_n - S_k < 0) \\ &= P(S_n \geq x\sqrt{n}) + \sum_{1\leq k\leq n} (1/2)P(A_k) \\ &= P(S_n \geq x\sqrt{n}) + (1/2)P(\cup_{1\leq k\leq n} A_k) \\ &= P(S_n \geq x\sqrt{n}) + (1/2)P(\max_{1\leq k\leq n} S_k \geq x\sqrt{n}) \end{aligned}$$

Hence

$$P(\max_{1\leq k\leq n} S_k \geq x\sqrt{n}) \leq 2P(S_n \geq x\sqrt{n})$$

and

$$\begin{aligned} P(\max_{1\leq k\leq n} |S_k| \geq x\sqrt{n}) &\leq P(\max_{1\leq k\leq n} S_k \geq x\sqrt{n}) + P(\max_{1\leq k\leq n} (-S_k) \geq x\sqrt{n}) \\ &\leq 4P(S_n \geq x\sqrt{n}) \\ &= \frac{4}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt \\ &\leq 2\int_x^\infty (t/x) e^{-t^2/2} dt = \frac{2}{x} e^{-x^2/2} \end{aligned}$$

(ii) **(9 points)**

Upper bound (5 points)

Let

$$T_n = \frac{\max_{1\le i\le n}|S_i|}{(2n\ln\ln n)^{1/2}}$$

For fixed $\theta > 1$, let $n_k = [\theta^k]$. For $\varepsilon > 0$, by (i)

$$P(T_{n_k} \ge 1+\varepsilon) \le C\exp(-(1+\varepsilon)^2\ln\ln n_k) \le C(\ln n_k)^{-(1+\varepsilon)^2},$$

which is summable. Hence by the Borel-Cantelli lemma

$$\limsup_{k\to\infty} T_{n_k} \le 1 \ a.s.$$

Now m large, choose k so that $n_{k-1} < m \le n_k$. Then

$$\limsup_{m\to\infty} T_m \le \limsup_{k\to\infty} \theta^{1/2} T_{n_k} \le \theta^{1/2} \ a.s.$$

This proves that

$$\limsup_{m\to\infty} T_m \le 1 \quad a.s.$$

by the arbitrariness of $\theta > 1$.

Lower bound (4 points)

To prove the lower bound of lim sup, let $n_k = e^{k\ln k}$. Then

$$\begin{aligned}
&\limsup_{k\to\infty}\frac{S_{n_k}}{(2n_k\ln\ln n_k)^{1/2}}\\
\ge\ &\limsup_{k\to\infty}\frac{S_{n_k}-S_{n_{k-1}}}{(2n_k\ln\ln n_k)^{1/2}} - \limsup_{k\to\infty}\frac{|S_{n_{k-1}}|}{(2n_k\ln\ln n_k)^{1/2}}\\
=\ &\limsup_{k\to\infty}\frac{S_{n_k}-S_{n_{k-1}}}{(2n_k\ln\ln n_k)^{1/2}} - \limsup_{k\to\infty}\frac{\sqrt{n_{k-1}}}{\sqrt{n_k}}\frac{|S_{n_{k-1}}|}{(2n_{k-1}\ln\ln n_{k-1})^{1/2}}\\
=\ &\limsup_{k\to\infty}\frac{S_{n_k}-S_{n_{k-1}}}{(2(n_k-n_{k-1})\ln\ln n_k)^{1/2}}.
\end{aligned}$$

It is known that for a standard normal random variable Z

$$P(Z > x) \approx \frac{1}{x}e^{-x^2/2} \quad \text{as} \quad x\to\infty$$

Hence for any $0 < \varepsilon < 1/2$ and k large enough

$$\begin{aligned}
P\left(\frac{S_{n_k}-S_{n_{k-1}}}{(2(n_k-n_{k-1})\ln\ln n_k)^{1/2}} > (1-\varepsilon)\right) &\approx \frac{1}{\ln\ln n_k}\exp(-(1-\varepsilon)^2\ln\ln n_k)\\
&\approx \frac{1}{(\ln k)(k\ln k)^{(1-\varepsilon)^2}}
\end{aligned}$$

which is not summable. Note that $\{S_{n_k}-S_{n_{k-1}}, k\ge 1\}$ are independent. By the Borel-Cantelli lemma,

$$\limsup_{k\to\infty}\frac{S_{n_k}-S_{n_{k-1}}}{(2(n_k-n_{k-1})\ln\ln n_k)^{1/2}} \ge 1 \ a.s.$$

This proves the lower bound by the inequalities above.