ASSOCIATED PRIMES OVER ORE EXTENSIONS AND GENERALIZED WEYL ALGEBRAS

by

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We describe those ideals of an arbitrary associative unital ring R which extend to prime ideals in three noncommutative extensions. The first extension is the polynomial extension $S = R[x; \sigma]$ invented by Ore. In the case when σ is surjective, we completely characterize those ideals I of R for which IS is a prime ideal. We do the same for the closely related skew Laurent-polynomial extensions $R[x, x^{-1}; \sigma]$. Our results include specialization to Noetherian base rings, when these ideals are closely related to prime ideals of R. In addition, we generalize a well known result from commutative algebra regarding the homogeneity of associated prime ideals in $\mathbb{Z}$-graded rings. The last extensions, invented by Bavula, are known as generalized Weyl algebras. These extensions are closely related to the previous extensions. We characterize those ideals of R which extend to ideals of A in several surprising ways.

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CHAPTER I

INTRODUCTION

I.1. Motivation and Background

The past decade has seen a large increase in interest regarding noncommutative algebraic structures. Some of this growth is due to the study of many new noncommutative objects and their relation to quantum geometry and physics. However, many noncommutative notions that were developed earlier in the century are now being looked at again using modern tools. That is the spirit of this dissertation. Take the Fundamental Theorem of Arithmetic. This ancient theorem states that every positive integer is the unique product of powers of prime integers. Commutative algebraists generalized the notion of prime integers to prime ideals; simply put, a prime ideal in a commutative ring is an ideal which, when it contains a product of elements, contains one of the factors. This led to an important decomposition theory in classical commutative algebra: the generalization of the Fundamental Theorem of Arithmetic commonly known as primary decomposition. But early noncommutative algebraists realized they could take the definition of a prime ideal a step further. A prime ideal, in the most general sense, is an ideal which, when it contains a product of ideals, contains one of the factors.

Unfortunately, primary decomposition–writing ideals and modules as intersections of 'primary' ideals and submodules–along with the notion of the corresponding associated prime ideals, does not generalize to the noncommutative setting. However, prime modules–nonzero modules whose annihilator is constant over all nonzero submodules–and their associated prime ideals are still useful for describing structural relationships over noncommutative rings. For instance, it is easy to see that over a Noetherian ring, every finitely-generated module, M, will have maximal prime submodules, the direct sum of which amounts to a 'prime socle' which is essential, or large, in M. The theory of affiliated series and the corresponding affiliated primes for modules over noncommutative rings provides much of the structural information that primary decomposition yields in the commutative case.

Associated primes are also an elementary object in commutative geometry. Let R be a commutative, associative, unital ring. The collection of prime ideals of R, known as the prime spectrum of R, is a scheme. Any module, M, can be considered as a sheaf over $\mathrm{Spec}(R)$. The geometric support of the module consists of the set of primes, P, for which the localizations, M_P, are nonzero. The associated primes are the minimal elements of the support. Again, in the noncommutative setting this notion does not fully generalize. Prime ideals are no longer the objects of interest in noncommutative localization; the notion of support for a module requires more machinery. However, M can still be considered a sheaf over $\mathrm{Spec}(R)$, so obtaining information about the set of prime ideals of a ring is still very useful in the noncommutative setting. Moreover, having information about Spec(R) can help in the abstract process of noncommutative localizaton.

We take these two notions from commutative algebra and geometry as motivation for the main work of this dissertation. Our primary focus is to compute the associated primes of so-called induced modules over noncommutative extensions of an arbitrary associative unital ring, R, using only information about the base ring and module, and the data needed to construct the extensions.

Our initial inquiry is a study of the noncommutative polynomial extensions invented by Ore in the early 1930's. This work is loosely tied to the problem of determining the primes of a ring, R, which extend to the commutative polynomial extension $R[x]$. It is interesting to note that the commutative result showing that, for each prime $P < R$, the ideal $P[x] < R[x]$ is prime was not proven until the early 1970's by Brewer and Heinzer [8]. Although more recent proofs by Faith [13] offered techniques which could be used for noncommutative rings, the archetypal work for the noncommutative extensions that are central to this dissertation was done in 1979 by Irving in [15]. We have taken Irving's work as the primary inspiration for what is done in Chapter 2, although more recent work has provided more convenient techniques for describing prime ideals.

For the second of our investigations we characterize those ideals which extend in a meaningful way to prime ideals in the rings invented by Bavula in the early 1990's [3]. These rings are generalized versions of Weyl algebras, and hence, closely allied with our previous work. However, it is surprising that the results concerning Ore extensions can be extended to help calculate prime ideals in these rings. The representation theory of these rings is well known, and has been studied both directly, by Bavula, Drozd, Gusner, and Ovsienko [11], and indirectly through even more general noncommutative extensions called down-up algebras, by Kulkarni [16], Carvalho and Musson [9], and most recently by Cassidy and Shelton in [10]. However, characterization of the prime ideals of these rings has not been attempted until now. It is very possible that the work done in Chapter 3 will shed some light on the geometry of modules over these rings.

I.2. Ore Extensions

First, we study prime ideals in Ore extensions [18]. These extensions cover a large class of noncommutative polynomial extensions. Our work is concerned with a subclass of these rings, the Ore extensions of the form $S = R[x;\sigma]$. These rings are defined by turning the free left R-module on basis $1, x, x^2, \ldots$ into a ring by imposing the defining relation $xr = \sigma(r)x$, where σ is an endomorphism of the base ring. A natural question that arises from this construction is how the ideals of S can be built from those of R. Our work investigates the rings $S = R[x;\sigma]$ when σ is surjective.

In this case, $P + Sx$ is always a prime ideal of S, so we shall restrict our attention to more interesting extensions of ideals of R which are prime in S. Specifically, our work concerns whether there is a way of characterizing ideals, $I \leq R$, for which SIS is a prime ideal of S. Let us call such ideals σ-*associated* ideals.

The theories regarding prime ideals and noncommutative geometry were in their infancy when Irving first attempted to answer this question. In [15] he first considered the case when R is a commutative ring and σ is any endomorphism of R. Irving defined a σ-*invariant* ideal to be an ideal of R for which $I = \sigma^{-1}(I)$. He then defined a σ-*prime* ideal to be a σ-invariant ideal, I, for which, given ideals $J, K \leq R$, with $\sigma(J) \subseteq J$ and $JK \subseteq I$, one can conclude either $J \subseteq I$ or $K \subseteq I$. He then showed that these were precisely the σ-associated ideals of the commutative ring R. However, his results do not extend to arbitrary rings.

Subsequent inequivalent definitions for σ-prime ideals have been made in [19] and [17]. None of these definitions completely characterize σ-associated ideals without imposing other hypotheses. Irving did notice that such an ideal, I, must be σ-invariant. This turns out to be a crucial observation. It is one that has been overlooked by subsequent studies, many of which focus on a less stringent requirement that I be a σ-ideal, that is, $\sigma(I) \subseteq I$.

The most recent attempt to characterize σ-associated ideals was by S. Annin in [1]. Annin realized that it was much easier to find such ideals by observing that each arose naturally via the associated prime of the induced skew-polynomial module

$M[x;\sigma] := M \otimes_R S$ for a certain right R-module, M. His work was limited to showing when an associated prime $\mathfrak{p}$ of M extends as $\mathfrak{p}[x;\sigma]$ to an associated prime of $M[x;\sigma]$. However, Annin made several overly restrictive assumptions. His characterization of such modules, which he terms σ-*compatible*, required that:

$$mr = 0 \text{ if and only if } m\sigma(r) = 0, \text{ for all } m \in M, r \in R.$$

We note that this is the same as requiring that $\text{ann}(m)$ be a right σ-invariant σ-ideal for every nonzero $m \in M_R$. This condition is too strong and more in the spirit of commutative algebra than noncommutative algebra.

We will show, when σ is surjective, the associated primes of M extend to associated primes of the induced module whenever the annihilators of nonzero submodules of M are σ-invariant ideals. When σ is not onto, it becomes clear that the required data for computing the associated primes of $M[x;\sigma]$ are no longer contained in the module M.

In Chapter II, we completely characterize the ideals, $I \leq R$, whose polynomial extensions, $I[x;\sigma]$, are prime ideals of the Ore extension $R[x;\sigma]$ when σ is surjective. In some sense, our work takes the ideas previously laid out and formulates a full and coherent theory in as general a setting as possible. Although Annin's work on the subject is incomplete, his original intent in characterizing prime ideals via associated primes is quite useful to our end. To see this, note that if $P \leq R$ is prime, R/P is always a prime module with associated prime P. As corollaries to our main theorem in Chapter II, we are able to describe more precisely the ideals which extend to prime ideals when working over a Noetherian ring. In the process of proving our main result we prove that associated primes of $\mathbb{Z}$-graded modules over an arbitrary $\mathbb{Z}$-graded ring are homogeneous ideals. This fully generalizes the commutative version of this result, as well as simplifies the method of proof for our main results.

Also in the second chapter, we discuss the ideals of R whose polynomial extensions are prime in the skew-Laurent extension $R[x, x^{-1}; \sigma]$. Given an automorphism, σ of R, the ring $R[x, x^{-1}; \sigma]$ is defined analogously to the Ore extension. The characterization of the ideals $I < R$ for which $I[x, x^{-1}; \sigma]$ is prime has been well studied in [17]. The fundamental result is that these ideals are σ-prime in the sense of the authors. We show our work with Ore extensions easily generalizes to the skew-Laurent polynomial rings and give a new description of the σ-prime ideals of R, which we call *Laurent σ-associated* ideals.

I.3. Generalized Weyl Algebras

There is a class of rings closely related to Ore extensions, invented by Bavula in [3]. Given a base ring, R, an automorphism, σ, of R and a regular, central element, q, we construct the free left R-module $A = R[d, u; \sigma, q]$ on the basis $\{\dots, d^2, d, 1, u,^2, \dots\}$. Then A can be made into a ring by imposing the following relations:

$$dr = \sigma^{-1}(r)d,\ ur = \sigma(r)u,\ du = q,\ ud = \sigma(q).$$

The ring A is called a *generalized Weyl algebra*, with some abuse noted to term such objects 'algebras.' To see how these rings relate to Ore extensions requires only some minor observations on their construction. First, it is easier to see that A is a ring by identifying it with the subring of the skew-Laurent extension $R[u, u^{-1}; \sigma]$ generated by R, u and $d = qu^{-1}$. Then, in fact, $R[u; \sigma] \subset A \subseteq R[u, u^{-1}; \sigma]$. Thus, A naturally inherits a $\mathbb{Z}$-grading from $R[u, u^{-1}; \sigma]$. Any right R-module, M, can be extended to a graded A-module by taking $M \otimes_R A$. Therefore we can exploit Proposition II.3.1 of Chapter II. Moreover, A is an overring for $R[u; \sigma]$, so it is natural to ask how the theory regarding Ore and skew-Laurent extensions described in Chapter II relates to the primes of the generalized Weyl algebra A.

In Chapter III, we characterize the associated primes of the induced A-module of an arbitrary module M_R. In doing so, we show that to each prime submodule, $\mathcal{N}$, of $M \otimes_R A$, there is a corresponding Laurent σ-associated ideal $I < R$. Moreover, our main result provides a formulaic approach to computing the associated prime of $\mathcal{N}$ from this Laurent σ-associated ideal. We note that the prime extensions of the Laurent σ-associated ideal, I, are not necessarily of the form IA. As corollaries to our main results, we compute the associated primes of the induced module in the special cases when R is Noetherian and when σ has finite order. Our approach is very much the same as in the case of the Ore and skew-Laurent extensions. Some of the machinery needed, although very similar to that found in Chapter II, is much stronger. The coefficients which arise from multiplying powers of d and u require more information be extracted from the module M. We note, as a consequence of our main theorem, that certain primes arise from d^k and u^k-torsion modules over A. These prime ideals appear to agree with known information about finite-dimensional simple modules over A.

I.4. Summary of results

Some readers will note that this dissertation does not follow the usual outline. We have intentionally kept the theories regarding different rings compartmentalized by chapter.

Chapter II is devoted to the results regarding the extension of ideals to prime ideals in the Ore extension $R[x;\sigma]$ and skew-Laurent extension $R[x,x^{-1};\sigma]$. This includes a complete characterization of such ideals when σ is surjective in the former extensions, as well as results regarding the associated primes of induced modules. Other than the preliminary definitions given here, the necessary theory is contained within the chapter. We state the main results, offer several simple examples, and then prove the main results.

Chapter III consists of the material on extensions to primes in generalized Weyl algebras. We use several of the definitions set out in the previous section and those in Section II.1, as well as Proposition II.3.1. After stating the main result of the chapter we again give examples that outline some of the nuances of the theory and then prove those results.

Chapter II is published under the same title in the Journal of Algebra.

CHAPTER II

ASSOCIATED PRIMES OVER ORE EXTENSIONS

II.1. Preliminaries and Statements of Theorems

Let R be a ring with identity and let σ be an endomorphism of R. Consider $S = R[x;\sigma]$, the left Ore extension. We use the convention that coefficients are written on the left and the defining relation is $xr = \sigma(r)x$ [1, 14]. One question which arises in this construction is how the prime ideals of $R[x;\sigma]$ are built from ideals of R. Much of the initial work regarding the propagation of primes when R is commutative was done by Irving in [15]. The technique pioneered by Irving and modeled by others was to first define the notion of a 'σ-prime' ideal. This led to several different inequivalent definitions, all of which are based on the usual noncommutative definitions for prime ideals. Conditions were then given under which, given a σ-prime ideal, $I \leq R$, one can conclude IS is prime. We have chosen a slightly different tack. This work grew out of an attempt to place a more noncommutative framework on previous work, [1], which related the associated prime ideals of a right R-module, M, to the associated primes of the induced module $M[x,\sigma]_S$. We quickly realized this relationship was greatly simplified by assuming that σ is surjective. Indeed, with that hypothesis, results relating the set of annihilator ideals of M_R and the set of associated primes of $M[x;\sigma]_S$ can then be read off easily. Using these results, one is much better off simply computing the associated primes of $M[x,\sigma]$ directly. As a corollary to our main result, we show that an ideal I for which $I[x;\sigma]$ is prime is precisely an ideal we define as the σ-associated ideal to some σ-prime module N_R.

In this section we provide the definitions and statements of the main result and its corollaries. In the second section, we discuss several examples outlining the use of these results. The last section is devoted to the proofs of several preliminary results and then the proofs of the principal results.

With minimal notation we can state our main result. We recall, in general, that a nonzero submodule $N < M$ is *prime* if $\operatorname{ann}(N')$ is constant across all nonzero submodules of N and in such cases, $\operatorname{ann}(N)$ is necessarily a prime ideal. Also a left, right or two-sided ideal I is a σ-*ideal* if $\sigma(I) \subseteq I$ [14], and is called a σ-*invariant* ideal if $I = \sigma^{-1}(I)$ [15].

Definition II.1.1. For any subset $I \subseteq R$, let $I_\sigma = \bigcap_{j\in\mathbb{N}} \sigma^{-j}(I)$. We say that a nonzero submodule $N < M$ is a σ-*prime* submodule if $(\operatorname{ann}(N'))_\sigma$ is constant over nonzero submodules of N and additionally $\sigma^{-1}((\operatorname{ann}(N))_\sigma) \subseteq (\operatorname{ann}(N))_\sigma$.

We note that neither $\operatorname{ann}(N)$ nor $(\operatorname{ann}(N))_\sigma$ need be prime for a σ-prime submodule N. Nevertheless, when N is σ-prime we refer to $I = (\operatorname{ann}(N))_\sigma$ as a σ-*associated* ideal of M and let $\sigma\text{-}\operatorname{Ass}(M)$ denote the set of σ-associated ideals. If $I \in \sigma\text{-}\operatorname{Ass}(M)$, by definition $\sigma^{-1}(I) \subseteq I$. Moreover, $I = (\text{ann}(N))_\sigma$ for some σ-prime submodule N, so $I = \bigcap_{j \in \mathbb{N}} \sigma^{-j}(\text{ann}(N)) \subseteq \bigcap_{j>0} \sigma^{-j}(\text{ann}(N)) = \sigma^{-1}((\text{ann}(N))_\sigma) = \sigma^{-1}(I)$. Thus a σ-associated ideal is σ-invariant.

If I is a subset of R we write $I[x;\sigma]$ for the set of polynomials in $R[x;\sigma]$ whose (left) coefficients are all in I. Even if I is an ideal of R, $I[x,\sigma]$ need not be an ideal in $R[x;\sigma]$.

Theorem II.1.2. *Let R be a ring with identity and let σ be a surjective endomorphism. For any right R-module M,* $\text{Ass}(M[x;\sigma]) = \{I[x;\sigma] \mid I \in \sigma\text{-}\text{Ass}(M)\}$.

It is apparent that $I[x,\sigma]$ can be an associated prime of $M[x,\sigma]$ when I is not a prime of R. More remarkably, I need not be the annihilator of a submodule of M, as illustrated in Example II.2.1. However, the following corollary shows that σ-associated ideals are precisely those ideals which extend to prime ideals.

Corollary II.1.3. *Suppose σ is surjective. Then the following are equivalent conditions on an ideal $I \leq R$:*

1. *I is the σ-associated ideal to some σ-prime module N_R;*

2. *$I[x;\sigma]$ is a prime ideal of S.*

Recall that a module is *σ-compatible* if all annihilators of elements of M are σ-invariant σ-ideals. This was shown to be a sufficient condition to conclude that the associated primes of $M[x;\sigma]$ are precisely the extensions of the associated primes of

M [1]. However, we observe that any σ-invariant associated prime is automatically a σ-associated ideal. Consequently, the associated primes of $M[x;\sigma]$ coincide with the extensions of the associated primes of M precisely when every associated prime of M is σ-invariant and every other annihilator ideal I is either not σ-invariant, or satisfies the condition that for every submodule N with $\text{ann}(N) = I$, there exists $0 \neq K < N$ such that $(\text{ann}(K))_\sigma \neq I$. In light of this, the analogue of the main result of [1], is clear:

Corollary II.1.4. *Suppose σ is surjective and M is a right R-module.*

1. *If $\mathfrak{p} \in \text{Ass}(M)$, then $\mathfrak{p}[x;\sigma] \in \text{Ass}(M[x;\sigma]_S)$ if and only if $\mathfrak{p}$ is σ-invariant;*

2. *If M is σ-compatible, or more generally, if every annihilator of a submodule of M is σ-invariant, then $\text{Ass}(M[x;\sigma]_S) = \{\mathfrak{p}[x;\sigma] \mid \mathfrak{p} \in \text{Ass}(M)\}$.*

When R is Noetherian, σ is an automorphism, and we get a much stronger result:

Corollary II.1.5. *If R is a Noetherian ring, then $\text{Ass}(M[x;\sigma]_S) = \{\mathfrak{p}_\sigma[x;\sigma] \mid \mathfrak{p} \in \text{Ass}(M)\}$.*

Remarks II.1.6. Results parallel to Theorem II.1.2 and its corollaries for the skew-Laurent polynomial extensions are made by altering Definition II.1.1. In order to define $R[x,x^{-1};\sigma]$, σ must be an automorphism. When σ is an automorphism and $I \subseteq R$, define $I_{\sigma^*} = \bigcap_{j\in\mathbb{Z}} \sigma^j(I)$. We call a nonzero submodule $N \leq M$ *Laurent σ-prime* if $(\text{ann}(N'))_{\sigma^*}$ is constant over all nonzero submodules of N and call an ideal, I, a *Laurent σ-associated ideal* of M if $I = (\mathrm{ann}(N))_{\sigma^*}$ for some Laurent σ-prime submodule N. We will denote the set of Laurent σ-associated ideals of M by σ^*- $\mathrm{Ass}(M)$. Although this altered definition only applies for the skew-Laurent extensions, the Laurent polynomial version of Theorem II.1.2 is now easily deduced: $\mathrm{Ass}(M[x,x^{-1};\sigma]_{R[x,x^{-1};\sigma]}) = \{I[x,x^{-1};\sigma] \mid I \in \sigma^*\text{- Ass}(M)\}$. The corollaries of this are also straightforward. Observe that for any left, right, or two-sided ideal I, I_{σ^*} is σ-invariant. Thus for every $\mathfrak{p} \in \mathrm{Ass}(M)$, $\mathfrak{p}_{\sigma^*}$ is automatically a Laurent σ-associated ideal of M. Therefore $\{\mathfrak{p}_{\sigma^*}[x,x^{-1};\sigma] \mid \mathfrak{p} \in \mathrm{Ass}(M)\} \subseteq \mathrm{Ass}(M[x,x^{-1};\sigma]_{R[x,x^{-1};\sigma]})$. Moreover, the notion of a σ-prime ideal in the Laurent extension case is well established. A σ-prime ideal is a σ-invariant ideal, P, which satisfies the condition that if I, J are σ-invariant ideals with $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$. Such ideals always extend to prime ideals of $R[x,x^{-1};\sigma]$ [17]. In particular, the analogue of Corollary II.1.3 is that an ideal is σ-prime if and only if it is the Laurent σ-associated ideal of some Laurent σ-prime module. The proofs for the results for the Laurent extensions are similar to the proofs of the main result that appear in Section II.3 and are therefore omitted.

II.2. Examples

A module can easily fail to be σ-compatible, as defined in [1], but still have each associated prime of $M[x;\sigma]$ be extended from one of M. For example, if R is any simple ring with automorphism σ, then II.1.2 shows that for any nontrivial module M, $\{(0)\} = \text{Ass}(M[x;\sigma]_S) = \{\mathfrak{p}_\sigma[x;\sigma] \mid \mathfrak{p} \in \text{Ass}(M)\}$. However, if R has a proper nonzero right ideal J which is not σ-invariant, then $M = R/J$ is not σ-compatible.

It is more interesting, in light of Theorem II.1.2, to investigate modules, M, for which the associated primes of M fail to extend. The first of these investigations involves an associated prime which is not σ-invariant. Throughout the next examples we let k denote a field.

Example II.2.1. Let $R = k[s,t]$ and $M_R = R/(t)$. Let σ be the k-algebra automorphism of R transposing s and t. Clearly $\text{Ass}(M_R) = \{(t)\}$. But (t) is not σ-invariant since $\sigma^{-1}((t)) = (s)$. Now $(t)_\sigma = (t) \cap (s) = (st)$. We observe that $(st) \in \sigma\text{-Ass}(M)$ and so by II.1.2, $\text{Ass}(M[x;\sigma]) = \{(st)[x;\sigma]\}$. Note that (st) is not prime, and more, is not the annihilator of any submodule of M.

For $\mathfrak{p}[x;\sigma] \in \text{Ass}(M[x;\sigma]_S)$, $\mathfrak{p}$ need not be a prime ideal of R. However, in the above example $(t)_\sigma = (st)$, so one question that arises is whether or not $\mathfrak{p}_\sigma$ is a σ-associated ideal when $\mathfrak{p}$ is prime. The following example shows that it need not be, even when R is commutative. Note that by II.1.5 we must begin with a non-Noetherian base ring.

Example II.2.2. Let $R = k[\dots, t_{-1}, t_0, t_1, \dots]$, and $M_R = R/(\dots, t_{-1}, t_0)$. Consider the k-algebra automorphism of R given by $\sigma(t_i) = t_{i-1}$ for all i. Clearly M is prime with annihilator $(\dots, t_{-1}, t_0)$. Thus $\operatorname{Ass}(M) = \{(\dots, t_{-1}, t_0)\}$. Observe that $(\dots, t_{-1}, t_0)_\sigma = (\dots, t_{-1}, t_0) \cap (\dots, t_{-1}, t_0, t_1) \cap (\dots, t_{-1}, t_0, t_1, t_2) \cap \dots = (\dots, t_{-1}, t_0)$, but $(\dots, t_{-1}, t_0)$ is not σ-invariant, hence not a σ-associated ideal. Therefore by Theorem II.1.2, $\operatorname{Ass}(M[x;\sigma]_S) = \emptyset$. Thus an associated prime of M_R need not extend in any meaningful way to an associated prime of $M[x;\sigma]_S$.

The next example illustrates that a non-prime annihilator I can be a σ-associated ideal which is not $\mathfrak{p}_\sigma$ for any associated prime $\mathfrak{p}$.

Example II.2.3. Let $R = k[\dots, t_{-2}, t_{-1}, t_0, t_1, t_2, \dots]/(t_i^2)$, and define $\bar{t}_i = t_i + (t_i^2)$. Set $M_R = R_R$ and let σ be the k-algebra automorphism of R given by $\sigma(\bar{t}_i) = \bar{t}_{i-1}$ for all i. We claim that M is σ-prime, but note that it is not prime. Observe that $\operatorname{ann}(M) = 0$ is σ-invariant. Thus $(\operatorname{ann}(M))_\sigma = 0$. In order to show M is σ-prime it will be enough to show $(g)_\sigma = 0$ for any $g \in (\bar{t}_i)_{i\in\mathbb{Z}}$. If $g \in (\bar{t}_i)_{i\in\mathbb{Z}}$, then there exist $j_1, j_2, \dots, j_n \in \mathbb{Z}$ such that $g \in (\bar{t}_{j_1}, \bar{t}_{j_2}, \dots, \bar{t}_{j_n})$. Now $(g)_\sigma \subseteq (\bar{t}_{j_1}, \bar{t}_{j_2}, \dots, \bar{t}_{j_n})_\sigma = 0$. Therefore M is σ-prime. Thus 0 is the only σ-prime ideal of M. Therefore by Theorem II.1.2, $\operatorname{Ass}(M[x;\sigma]) = \{0\}$. In contrast, we observe that every nonzero cyclic submodule $fR \leq M$ contains a nonzero cyclic submodule whose annihilator strictly contains $\operatorname{ann}(fR)$. That is, M has no cyclic prime submodules, hence no prime submodules. Therefore $\operatorname{Ass}(M) = \emptyset$.

II.3. Proofs of the Main Results

The proof of Theorem II.1.2 relies on some elementary initial results. The first result is well known in commutative algebra. We generalize the result found in [12].

Proposition II.3.1. *If* $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i$ *is* $\mathbb{Z}$*-graded ring with identity,* $\mathcal{M}_A = \bigoplus_{i \in \mathbb{Z}} \mathcal{M}_i$ *is a graded module,* $\mathcal{N} \leq \mathcal{M}$ *is a prime submodule, and* $\mathfrak{q} = \operatorname{ann}(\mathcal{N})$*, then* $\mathfrak{q}$ *is a homogeneous ideal.*

Proof: Let $a = a_0 + \cdots + a_k \in \operatorname{ann}(\mathcal{N})$, where each a_i is a nonzero element of $\mathcal{A}_{m_i}$ for some integers $m_0 < \cdots < m_k$. It will be enough to show that $a_0 \in \operatorname{ann}(\mathcal{N})$. It will then follow by induction on k that $\mathcal{N} a_i = 0$ for each i, and so the homogeneous terms of a belong to $\operatorname{ann}(\mathcal{N})$.

Let $m \in \mathcal{N}$ be an element of least possible length. That is, every element is the unique sum of nonzero homogeneous elements, and for m, it involves the least number of terms possible among elements of $\mathcal{N}$. Write $m = m_0 + \cdots + m_l$, where each m_i is a nonzero element of M_{n_i} for some integers $n_0 < \cdots < n_l$. Clearly, for any homogeneous component, $\mathcal{A}_r$, every nonzero element of $m\mathcal{A}_r$ has length l. However, a_0 annihilates the first term of every nonzero element of $m\mathcal{A}_r$, hence every nonzero element of $m\mathcal{A}_r a_0$ has length less than l. By the minimality of l, it must be that $m\mathcal{A}_r a_0 = 0$. Thus $m\mathcal{A}a_0 = 0$. As $\mathcal{N}$ is prime, $a_0 \in \operatorname{ann}(m\mathcal{A}) = \operatorname{ann}(\mathcal{N})$. □

Corollary II.3.2. *If* $\mathfrak{q} \in \operatorname{Ass}(M[x;\sigma]_S)$*, then* $\mathfrak{q} = I[x;\sigma]$ *for some* σ*-invariant ideal* $I \leq R$*.*

Proof: We grade $S = R[x;\sigma]$ and $M[x;\sigma]_S$ by degree in x. The preceding proposition shows $\mathfrak{q}$ is homogeneous with respect to this grading. Since $M[x;\sigma]$ is x-torsion-free, it follows that $\mathfrak{q} = I[x;\sigma]$ for some ideal I. To show I is σ-invariant, let $\mathcal{N} \leq M[x;\sigma]$ be prime with annihilator $I[x;\sigma]$. On one hand, $0 \neq \mathcal{N}x \leq \mathcal{N}$, so $0 = \mathcal{N}xI = \mathcal{N}\sigma(I)x$. Thus $\sigma(I) \subseteq I$, which says $I \subseteq \sigma^{-1}(I)$. On the other, $0 = \mathcal{N}Ix \supseteq \mathcal{N}\sigma(\sigma^{-1}(I))x = Nx(\sigma^{-1}(I))$. Since $\mathcal{N}$ is prime, $\text{ann}(\mathcal{N}x) = I[x;\sigma]$. Consequently, $\sigma^{-1}(I) \subseteq I$. Therefore $\sigma^{-1}(I) = I$. □

Corollary II.3.3. *If σ is surjective and $N \leq M[x;\sigma]_S$ is prime, then* $\text{ann}_S(N) = I[x;\sigma]$*, where I is the σ-associated ideal of a σ-prime submodule of M.*

Proof: By the previous corollary, $\text{ann}_S(N) = I[x;\sigma]$, where $I \leq R$ is σ-invariant. Let $0 \neq f \in N$ be of minimal length l, and write $f = m_0x^{a_0} + \cdots + m_lx^{a_l}$, where each m_i is a nonzero element of N and $a_0 < \cdots < a_l$. We show m_0R is σ-prime with σ-associated ideal I.

Set $a = a_0$ and let $m \in m_0R$. Since σ is onto, we may select $r \in R$ so that $m_0\sigma^a(r) = m$. Let $J = \text{ann}_R(mR)$. Since $fS \subseteq N$ and $I \subseteq \text{ann}(N)$, $fSI = 0$, and so $mR\sigma^a(I) = 0$. As I is σ-invariant and σ is onto, $mRI = 0$. Thus $I \subseteq J$.

Observe that, for all $i \geq 0$, every nonzero element of $frRx^i$ has length l. Every element of $frRx^i(J_\sigma)$ has length less than l, so $frRx^iJ_\sigma = 0$, by the minimality of l. Since N is prime $J_\sigma S \subseteq \text{ann}_S(frS) = I[x;\sigma]$. Therefore $J_\sigma = I$ and we conclude m_0R is σ-prime. □

Lemma II.3.4. *For an R-module N,* $\operatorname{ann}_S(N[x;\sigma]) = (\operatorname{ann}_R(N))_\sigma[x;\sigma]$.

Proof: Since $N[x;\sigma]$ is homogeneous, $\operatorname{ann}_S(N[x;\sigma])$ is homogeneous. Let $rx^i \in \operatorname{ann}_S(N[x;\sigma])$. Then $Nx^jr = 0$ for all $j \geq 0$, or, equivalently, $r \in \sigma^{-j}(\operatorname{ann}_R(N))$ for all $j \geq 0$. That is, $rx^i \in \operatorname{ann}_S(N[x;\sigma])$ if and only if $r \in (\operatorname{ann}_R(N))_\sigma$. $\square$

Lemma II.3.5. *Let σ be surjective. Then N_R is σ-prime with σ-associated ideal I if and only if $N[x;\sigma]$ is prime with associated prime $I[x;\sigma]$.*

Proof: Suppose $N[x;\sigma]$ is prime with associated prime $I[x;\sigma]$. By Lemma II.3.4, $I = (\operatorname{ann}(N))_\sigma$. Let $m \in N$ and set $J = \operatorname{ann}_R(mR)$. Since $N[x;\sigma]$ is prime, $I[x;\sigma] = \operatorname{ann}_S((mR)[x;\sigma]) = J_\sigma[x;\sigma]$. Thus $J_\sigma = I$ and so N is σ-prime with σ-associated ideal I.

Conversely, if N is σ-prime with σ-associated ideal I, then II.3.4 shows $I[x;\sigma] = \operatorname{ann}_S(N[x;\sigma])$. If $N[x;\sigma]$ is not prime, then there exists a nonzero element $f \in N[x;\sigma]$ such that $J = \operatorname{ann}_S(fS)$ strictly contains $I[x;\sigma]$. Write $f = m_0x^{a_0} + \cdots + m_kx^{a_k}$, where $m_i \neq 0$ and $a_0 < \cdots < a_k$, and let J_0 be the set of constant coefficients from elements of J. Select a nonzero element $s \in J \setminus I[x;\sigma]$ of minimal length. Observe that if $s = r_0x^{b_0} + r_1x^{b_1} + \cdots + r_mx^{b_m} \in J$, then $r_0 + r_1x^{b_1-b_0} + \cdots + r_mx^{b_m-b_0} \in J$, as x acts without torsion. Thus $r_0 \in J_0$. Since s is of minimal length, $r_0 \notin I$ and so J_0 strictly contains I. However, every element of J_0 annihilates the term of lowest degree of every element of fS. In particular $m_0 x^{a_0} R x^j J_0 = 0$ for all $j \geq 0$. Thus $J_0 \subseteq \bigcap_{j \geq a} \sigma^{-j}(\text{ann}(m_0 R)) = \sigma^{-a_0}((\text{ann}(m_0 R))_\sigma) = \sigma^{-a_0}(I)$. Since I is σ-invariant, this implies $J_0 \subseteq I$, a contradiction. Therefore no such f exists, and $N[x;\sigma]$ is prime with associated prime $I[x;\sigma]$. $\square$.

We now have all of the preliminary results needed for the proof of the main result. The proof hinges on the fact that we already know the form of the associated primes.

Proof of Theorem II.1.2: If $\mathfrak{p} \in \text{Ass}(M[x;\sigma])$, then $\mathfrak{p} = \text{ann}(N)$ for some prime submodule $N \leq M[x;\sigma]$. By Corollary II.3.3, $\mathfrak{p} = I[x;\sigma]$, where I is the σ-associated ideal of a σ-prime submodule of M.

Conversely, if I is a σ-associated ideal of M, then $I = (\text{ann}(L))_\sigma$ for some σ-prime submodule $L \leq M$. By Lemma II.3.5, $I[x;\sigma] \in \text{Ass}(M[x;\sigma])$ as it is the annihilator of the prime submodule $L[x;\sigma] \leq M[x;\sigma]$. $\square$

Proof of Corollary II.1.3: If I is the σ-associated ideal to a σ-prime module, N_R, then II.3.5 shows $I[x;\sigma]$ is prime. Conversely, suppose $I[x;\sigma] \leq S$ is prime. Then $\sigma(I) \subseteq I$ since $xI[x;\sigma] \subseteq I[x;\sigma]$. So $I \subseteq \sigma^{-1}(I)$. As σ is surjective, $(xS)(\sigma^{-1}(I)S) = SIxS \subseteq I[x;\sigma]$. Since $I[x;\sigma]$ is prime, $\sigma^{-1}(I)S \subseteq I[x;\sigma]$. Thus $\sigma^{-1}(I) \subseteq I$. Therefore I is σ-invariant. According to II.3.5, it will be enough, to show $N = (R/I)[x;\sigma]$ is prime with associated prime $I[x;\sigma]$. Note that since I is σ-invariant, $N \cong S/(I[x;\sigma])$. Clearly, $I[x;\sigma] = \text{ann}_S(N)$. Let $f \in R[x;\sigma] \setminus I[x;\sigma]$ and set $J = \text{ann}_S((f + I[x;\sigma])S)$. Then $(f + I[x;\sigma])SJ \subseteq fSJ + I[x;\sigma]J \subseteq I[x;\sigma]$. So $fSJ \subseteq I[x;\sigma]$. Since $I[x;\sigma]$ is prime, $J \subseteq I[x;\sigma]$. Thus $J = I[x;\sigma]$ as required. $\square$

Proof of Corollary II.1.4: To verify (1), suppose $\mathfrak{p} \in \operatorname{Ass}(M)$ and let $N \leq M$ be a prime submodule with $\operatorname{ann}(N) = \mathfrak{p}$. Now N is automatically σ-prime with $\mathfrak{p}_\sigma$ as its σ-associated ideal, whenever $\mathfrak{p}_\sigma$ is σ-invariant. Consequently, if $\mathfrak{p}$ is σ-invariant, Lemma II.3.5 shows $\mathfrak{p}[x;\sigma] \in \operatorname{Ass}(M[x;\sigma])$. Conversely, if $\mathfrak{p}[x;\sigma] \in \operatorname{Ass}(M[x;\sigma])$, then II.3.2 shows that $\mathfrak{p}$ is a σ-associated ideal and is therefore σ-invariant.

For (2), we observe that the hypotheses along with (1) imply $\{\mathfrak{p}[x;\sigma] \mid \mathfrak{p} \in \operatorname{Ass}(M)\} \subseteq \operatorname{Ass}(M[x;\sigma])$. Suppose I is an ideal, $I \notin \operatorname{Ass}(M)$, but I is the annihilator of a nonzero submodule $N \leq M$. Since $I \notin \operatorname{Ass}(M)$, we may assume no such submodule is prime, and so contains a nonzero submodule L whose annihilator J strictly contains I. Since $I_\sigma = I$ and $J_\sigma = J$ by hypothesis, I cannot be a σ-associated ideal. This proves $\{\mathfrak{p}[x;\sigma] \mid \mathfrak{p} \in \operatorname{Ass}(M)\} = \operatorname{Ass}(M[x;\sigma])$. □

Proof of Corollary II.1.5: Suppose R is Noetherian and let $\mathfrak{q} \in \operatorname{Ass}(M[x;\sigma]_S)$. Then $\mathfrak{q} = I[x;\sigma]$ for some σ-prime ideal of M. In particular, there exists a σ-prime submodule $N \leq M$ with $I = (\operatorname{ann}(N))_\sigma$. Since R is Noetherian, there exists an ideal $\mathfrak{p}$ which is maximal among annihilators of nonzero submodules of N. We know $\mathfrak{p}$ is an associated prime of N, and hence of M. Moreover, since N is σ-prime, $\mathfrak{p}_\sigma = I$. Conversely, suppose $\mathfrak{p} \in \operatorname{Ass}(M)$ and let $L \leq M$ be a prime submodule with annihilator $\mathfrak{p}$. Set $I = \mathfrak{p}_\sigma$, and note that $\sigma(I) \subseteq I$. Since σ is an automorphism and R is Noetherian, this implies that I is σ-invariant. Therefore L is σ-prime with σ-associated ideal I. By II.3.4, $L[x;\sigma]$ is a prime submodule of $M[x;\sigma]$ with associated prime ideal, $I[x;\sigma]$. □

CHAPTER III

ASSOCIATED PRIMES OVER GENERALIZED WEYL ALGEBRAS

III.1. Preliminaries and Statements of Theorems

The main goal of this chapter is to characterize the associated primes of the induced module $\mathcal{M} = M \otimes_R A$ over the generalized Weyl algebra $A = R[d, u; \sigma, q]$, using only the data σ, q and the base module, M_R. Our first main result is that for any prime A-submodule $\mathcal{N} \leq \mathcal{M}$, there is a homogeneous prime submodule with the same annihilator. Following this, we show $I = \text{ann}_R(\mathcal{N})$ is a Laurent σ-associated ideal of R. These two results allow us to show that $J = \text{ann}_A(\mathcal{N})$ can be constructed by algorithm. For some associated primes, we can provide a formula using only $\text{ann}_R(\mathcal{N})$. The constructions show that an associated prime, J, of the induced module is not, in general, of the form IA, in contrast to the situation in Chapter II. When R is Notherian, we calculate $\text{Ass}(\mathcal{M})$ and show each associated prime J can be constructed based solely on information encoded in I. In the second section, we give several examples illustrating the use of these results. The final section is comprised entirely of the proofs of the main results.

We will be using the convention of writing all elements of A with R-coefficients on the left. We will also drop the tensor notation for elements of the induced module $\mathcal{M} = M \otimes_R A$, writing them as polynomials in d and u with left coefficients in M. For ease of notation it will be convenient to specify that $d^0 = u^0 = 1$. We record the following useful formulas for $k, j > 0$,

$$d^k u^j = \begin{cases} \sigma^{1-k}(q)\cdots\sigma^{j-k}(q)d^{k-j} & k \geq j \\ \sigma^{1-k}(q)\cdots q u^{j-k} & k < j, \end{cases}$$

and

$$u^k d^j = \begin{cases} \sigma^{k}(q)\cdots\sigma^{k-j+1}(q)u^{k-j} & k \geq j \\ \sigma^{k}(q)\cdots\sigma(q) d^{j-k} & k < j. \end{cases}$$

It is easier to write the R-coefficients using some notation. As in [3], for $i, j \geq 0$, we let $(-i, j)$ denote the coeffient in R of the product $d^i u^j$ and $(i, -j)$ that of $u^i d^j$ with respect to the standard basis $\{1, d^i, u^i \mid i > 0\}$.

Since σ is an automorphism we observe that if $J \leq R$ is any ideal for which $I = J_\sigma$ is σ-invariant, then $I = J_\sigma = \sigma(J_\sigma) = \sigma^2(J_\sigma) = \dots$. So, in fact, $I = J_{\sigma^*}$. Consequently, every σ-associated ideal is a Laurent σ-associated ideal. We note that the converse is not true. For example, if we recall the module M from Example II.2.2, we note that it is prime, and hence Laurent σ-prime, with 0 as its Laurent σ-associated ideal. However, as noted in the example, M has no σ-associated ideals. In particular 0 is not a σ-associated ideal of M.

Theorem III.1.1. *Suppose $\mathcal{N}_A$ is a prime submodule of $\mathcal{M}_A$. Then there exists a homogeneous prime submodule $\mathcal{L}_A$ of $\mathcal{M}$ such that* $\mathrm{ann}_A(\mathcal{N}) = \mathrm{ann}_A(\mathcal{L})$.

Via the above theorem, we may reduce to studying homogenous prime submodules of the induced module. We may further simplify to studying only cyclic homogeneous prime submodules. We will show that the M-coefficient of the generator of such a module in turn generates a Laurent σ-prime submodule of M_R. As a consequence, we have the following theorem.

Theorem III.1.2. *If $\mathcal{N}_A$ is a prime submodule of $\mathcal{M}_A$ and $J = \text{ann}_A(\mathcal{N})$, then $J \cap R$ is a Laurent σ-associated ideal of R.*

Corollary III.1.3. *There is a set map $\Phi : \text{Ass}(\mathcal{M}) \to \sigma^*\text{-}\,\text{Ass}(M)$ given by $\Phi(J) = J \cap R$.*

We note that, in general, Φ is not injective. In the next section we give an example (III.2.3) where $J \neq K \in \text{Ass}(\mathcal{M})$, but $J \cap R = K \cap R$. We conjecture that Φ is not necessarily surjective. Our goal is the explicit description of the fibers of Φ and, whenever possible, the image.

In the skew-Laurent extension $R[x, x^{-1}; \sigma]$, given $i \in \mathbb{Z}$ and a nonzero submodule $\mathcal{L} \leq M[x, x^{-1}; \sigma]$, we can always find an element of $\mathcal{L}$ with a homogeneous term of degree i. In general, this is not true for submodules of $\mathcal{M}$, as we shall see in Examples III.2.2 and III.2.3. In order to describe this phenomenon, we require some terminology. For any $0 \neq \mathcal{L} \leq \mathcal{M}$, if $\mathcal{L}$ is completely contained in $\text{span}\{d^i\}_{i>0}$ we will call it a *strictly negative* module; if $\mathcal{L}$ is completely contained in $\text{span}\{u^j\}_{j>0}$ we will call it *strictly positive*. Otherwise, we will say $\mathcal{L}$ is *manifest in all degrees*. Given any $J \in \text{Ass}(\mathcal{M})$, by definition, $J = \text{ann}(\mathcal{N})$ for some prime submodule $\mathcal{N}$. In the set of all prime submodules of $\mathcal{M}$ annihilated by J, there must be a prime submodule which is strictly positive or strictly negative, or barring those possibilities, a prime submodule which is manifest in all degrees and has no strictly positive or strictly negative submodules. Those three possibilities yield three formulas for constructing J from $I = J \cap R$. These formulas require the following standard notation: for $a \in r$ and $J \subseteq R$, the set $(a : J) = \{r \in R \mid ra \in J\}$ is commonly referred to as the left conductor of a into J. If a is central and J is a two-sided ideal, the left and right conductors coincide and it is easy to see that $(a : J)$ is a two-sided ideal.

Theorem III.1.4. *If $J \in \mathrm{Ass}(M_A)$ and $I = J \cap R$, then at least one of the following is true:*

1. $J = \sum_{k \leq 0} ((k, -k) : I) d^{-k} + \sum_{k > 0} I u^k$;
2. $J = \sum_{k < 0} I d^{-k} + \sum_{k \geq 0} ((k, -k) : I) u^k$;
3. $J = IA$.

If $I \in \sigma^*$-$Ass(M)$ is the R-annihilator of a strictly positive or strictly negative prime submodule of $\mathcal{M}_A$, then III.1.4 gives formulas for $J \in \mathrm{Ass}(\mathcal{M})$ with $\Phi(J) = I$ via parts 1 or 2, respectively. The remaining fibers of the Laurent σ-associated ideals in the image of Φ are given in 3. The three formulas given in Theorem III.1.4 are clearly left ideals of A. However, a quick check shows that they are indeed closed under the action of R, d and u on both sides, and are therefore two-sided ideals of A.

The case when R is Noetherian is easily deduced, since $\sigma^*\text{- Ass}(M) = \sigma\text{- Ass}(M) = \{\mathfrak{p}_\sigma \mid \mathfrak{p} \in \text{Ass}(M)\}$. This case is easier to describe using an indexing function:

$$gw : \sigma^*\text{- Ass}(M) \to \mathbb{N},$$

which we define via the following rules. Let $I \in \sigma^*\text{- Ass}(M)$. If $(k, -k) \in I$ for some $k > 0$, then $gw(I) = \min\{k > 0 \mid (k, -k) \in I\}$. Otherwise $gw(I) = 0$. We call $gw(I)$ the *gw-index* of I.

Corollary III.1.5. *If R is Noetherian, then Φ is surjective and* $\text{Ass}(\mathcal{M}_A) =$

$$\begin{aligned}
&\{\sum_{k\leq 0}((k,-k) : \mathfrak{p}_\sigma)d^{-k} + \sum_{k>0}\mathfrak{p}_\sigma u^k \mid \mathfrak{p} \in \text{Ass}(M),\ gw(\mathfrak{p}_\sigma) > 0\}\\
&\qquad\dot\cup\{\sum_{k<0}\mathfrak{p}_\sigma d^k + \sum_{k\geq 0}((k,-k) : \mathfrak{p}_\sigma)u^{-k} \mid \mathfrak{p} \in \text{Ass}(M),\ gw(\mathfrak{p}_\sigma) > 0\}\\
&\qquad\qquad\dot\cup\{\mathfrak{p}_\sigma A \mid \mathfrak{p} \in \text{Ass}(M),\ gw(\mathfrak{p}_\sigma) = 0\}.
\end{aligned}$$

In particular Φ is two-to-one over Laurent σ-associated ideals with positive gw-index and one-to-one over Laurent σ-associated ideals with zero gw-index.

This result for Noetherian rings is only slightly weakened if we instead consider the case when σ has finite order.

Corollary III.1.6. *If σ has finite order, then Φ is surjective and* $\text{Ass}(\mathcal{M}_A) =$

$$\begin{aligned}
&\{\sum_{k\leq 0}((k,-k) : I)d^{-k} + \sum_{k>0} Iu^k \mid I \in \sigma^*\text{- Ass}(M),\ gw(I) > 0\}\\
&\qquad\dot\cup\{\sum_{k<0} Id^k + \sum_{k\geq 0}((k,-k) : I)u^{-k} \mid I \in \sigma^*\text{- Ass}(M),\ gw(I) > 0\}\\
&\qquad\qquad\dot\cup\{IA \mid I \in \sigma^*\text{- Ass}(M),\ gw(I) = 0\}.
\end{aligned}$$

In particular Φ is two-to-one over Laurent σ-associated ideals with positive gw-index and one-to-one over Laurent σ-associated ideals with zero gw-index.

In both the Noetherian and finite order cases, a strictly positive or strictly negative prime submodule, $\mathcal{N}$ of $\mathcal{M}_A$ is d^k-torsion or u^k-torsion, respectively, where $k = gw(\text{ann}_R(\mathcal{N}))$. The gw-index is useful in the general case since it indicates when the coefficient ideal, $((k,-k):I)$, is in fact all of R. That is, the constructed ideal is the annihilator of a u^k- or d^k-torsion prime submodule of $\mathcal{M}_A$.

III.2. Examples

In this section we offer several examples which showcase some of the subtleties of Theorem III.1.4 and its corollaries. In each example, we give the data, R, M, σ, q, which yield the extension $A = R[d,u;\sigma,q]$, the corresponding induced module $M \otimes_R A$, and the map $\Phi : \text{Ass}(M \otimes_R A) \to \sigma^*\text{- Ass}(M)$. The first two examples are very simple, and reintroduce some very familiar rings in the context of generalized Weyl algebras. Throughout these examples, k is a field.

Example III.2.1. Let R be an arbitrary ring, $q = 1$, and take σ to be any automorphism of R. Then A is nothing more than the skew-Laurent extension $R[x,x^{-1};\sigma]$. By our work in Chapter II and our remarks at the beginning of this chapter, we know that for each prime ideal, $\mathfrak{p}$ of R, $\mathfrak{p}_\sigma$ is a Laurent σ-associated ideal. Theorem III.1.4 reasserts that $\mathfrak{p}_\sigma[x,x^{-1};\sigma]$ is a prime of $R[x,x^{-1};\sigma]$ for each prime $\mathfrak{p}$ of R, by taking $M = R/\mathfrak{p}$. More generally, given any Laurent σ-associated ideal I and its corresponding Laurent σ-prime module M, III.1.4 shows that $I[x,x^{-1};\sigma]$ is a prime ideal of $R[x,x^{-1};\sigma]$. That is, $\Phi : \text{Ass}(M \otimes_R A) \to \sigma^*\text{- Ass}(M)$ is a set isomorphism.

The above example indicates that Theorem III.1.4 covers the known results on skew-Laurent extensions. It also shows that Φ can be injective and in this case the fiber of each Laurent σ-associated ideals is a unique prime ideal of A. The next example shows that Φ need not be injective.

Example III.2.2. Let $R = k[s]$, $q = s$, and define σ to be the extension of the identity on k defined by $\sigma(s) = s - 1$. Then A is the first Weyl algebra over k. To see this, we use the usual construction of $A_1(k) = k[t, d/dt]$. The isomorphism is the k-algebra map $d/dt \mapsto d$, $t \mapsto u$. In the case that k is a field of characteristic zero, R has no nonzero σ-invariant ideals, and so the only Laurent σ-associated ideal is 0. Thus, 0 is the only ideal in the image of Φ; note that $gw(0) = 0$. If the characteristic of the field is $p > 0$, then $s, s-1, \cdots, s-p+1$ belong to a σ-cycle. That is, $I = ((s-p+1)\cdots(s-1)s)$ is a σ-invariant semiprime ideal of R with $gw(I) = p$. The R-module $M = R/(s)$ is prime and $I = (\text{ann}(M))_\sigma = (\text{ann}(M'))_{\sigma^{-1}})$. Therefore M is σ- and σ^{-1}-prime. Since A is Noetherian, Corollary III.1.5 states that I is in the image of Φ. In this case there are two ideals of the first Weyl algebra whose images under Φ are I. These ideals are

$$P = \cdots Rd^p + ((1-p, p-1) : I)d^{p-1} + \cdots + ((-2, 2) : I)d + I + Iu + Iu^2 + \cdots$$

$$= \cdots Rd^p + (s-1)d^{p-1} + \cdots + ((s-p+1)(s-p+2)\cdots(s-1))d + I + Iu + Iu^2$$

and

$$Q = \cdots Id^2 + Id + I + ((2, -2) : I)u + \cdots + ((p-1, 1-p) : I)u^{p-1} + Ru^p + \cdots$$

$$= \cdots Id^2 + Id + I + ((s-p+1)(s-p+2)\cdots(s-2)s)u + \cdots + (s)u^{p-1} + Ru^p + \cdots .$$

Thus we have constructed nontrivial prime ideals of the first Weyl algebra in nonzero characteristic. It is not true that $M \otimes A$ is prime. The module $M \otimes A$ contains the submodules $\sum_{i \geq p} M \otimes u^i$ and $\sum_{i \geq p} M \otimes d^i$, respectively, which are prime by Corollary III.3.19.

The above example shows that the submodules of the induced module that are prime can be relatively small, and can be d- or u-torsion. We can construct prime submodules of an induced module which are d^k or u^k-torsion for any k, and for which d^j or d^j are not in their annihilator for indices less than k, as shown in the next example.

Example III.2.3. Let $R = k[r, s, t]$, $q = r$, and σ be the k-algebra automorphism which sends r to s, s to t, and t to r. Set $M_R = R/(t)$ and let $\bar{u}^2 = 1 \otimes u^2 \in M \otimes_R A$. Note that M is prime and $I = (\text{ann}(M))_\sigma = (t)_\sigma = (rst)$, which is σ-invariant. Therefore M is σ-prime. According to Corollary III.3.19, the module $\bar{u}^2 A = M \otimes u^2 + M \otimes u^3 + \cdots$ is a prime submodule of $\mathcal{M}$. Corollary III.1.5 gives its A-annihilator:

$$P = \cdots + Rd^3 + ((-2, 2) : I)d^2 + ((-1, 1) : I)d + I + Iu^2 + \cdots$$

$$= \cdots + Rd^3 + (s)d^2 + (st)d + I + Iu^2 + \cdots .$$

One can clearly modify this example to get module which has a prime submodule of the induced module which is d^k-torsion for each $k > 0$.

A question that arises from the previous two examples is whether or not one can build a prime submodule of $M \otimes A$ which is completely contained in the span of u^i and d^i, but is d^j and u^j-torsion-free for all $j > 0$. The next example shows we can construct such a module. Corollary III.1.5 indicates that in our example we must work over a non-Noetherian ring.

Example III.2.4. Let $R = k[t_i]_{i\in\mathbb{Z}}$, $q = t_{-1}$, and σ be the k-algebra automorphism defined by $\sigma(t_i) = t_{i+1}$ for all $i \in \mathbb{Z}$. Set $M_R = R/(t_i)_{i\leq 0}$. Observe that M is prime and $((t_i)_{i\leq 0})_\sigma = 0$, which is σ-invariant. Thus M is σ-prime. Accordingly, $\mathcal{N} = M \otimes u + M \otimes u^2 + \cdots$ is a prime submodule of $M \otimes A$, by III.3.19. Its annihilator over A is $P = \cdots((-2, 2) : 0)d^2 + ((-1, 1) : 0)d + 0 + 0u + \cdots = 0$. Note that d kills $M \otimes u$, as $\sigma(q) \in \mathrm{ann}(M)$, but not all of $\mathcal{N}$ as $\sigma^i(q) \notin \mathrm{ann}(M)$ for $i > 1$.

Example III.2.5. If we take the same ring as in the previous example and let $M' = R$, then R is prime and its annihilator is 0. So M is easily seen to be Laurent σ-prime, and $M' \otimes_R A$ is prime with annihilator 0. If we take M as in the previous example and consider the module $M \oplus M'$, then 0 is a prime ideal of A and can be constructed by statements 1 and 3 of Theorem III.1.4. So it is easy to manufacture Laurent σ-associated ideals of R which are in the image of Φ, but whose fibers are not uniquely specified by the three constructions in III.1.4.

III.3. Proofs of the Main Results

In order to prove some of the main results we will have to first cover some intermediate results. Recall from Chapter II that we have already shown that associated primes of $\mathbb{Z}$-graded modules over a $\mathbb{Z}$-graded ring are homogeneous ideals. For an arbitrary module M_R, we consider a single prime submodule $\mathcal{N} \leq \mathcal{M} = M \otimes_R A$. We are thus concerned with characterizing elements of the form rd^j and ru^j which annihilate $\mathcal{N}$.

To begin, recall that every nonzero element of $\mathcal{M} = M \otimes_R A$ can be written uniquely as a finite sum of nonzero homogeneous elements. For any element $f \in \mathcal{M}$, the *length* of f will be the number of terms in this sum. Given a submodule $\mathcal{L} \leq \mathcal{M}$ we will call $0 \neq f \in \mathcal{L}$ an element of *minimal length* if no elements of $\mathcal{L}$ have length less than f. More generally, we say that f is of *weak minimal length* if f is of minimal length in the cyclic submodule fA. When we express a nonzero element as a sum of homogeneous terms, we will always assume each term is nonzero, unless stated otherwise.

Lemma III.3.1. *If f is of weak minimal length, then* $\mathrm{ann}_A(fA)$ *is homogeneous.*

Proof: Without loss of generality, let mu^k be the homogeneous term of least degree of f. An element $s \in A$ annihilates fA if and only if it annihilates fRd^i and fRu^i for all $i \geq 0$. As in the proof of II.3.1, since f is of minimal length, this is precisely when s annihilates mu^kRd^i and mu^kRu^i for all $i \geq 0$. Thus, s annihilates

mu^kA. Since mu^kA is homogeneous, its annihilator is homogeneous. Therefore each homogeneous term of s belongs to $\text{ann}_A(mA)$. But then each term of s belongs to $\text{ann}_A(fA)$, and so $\text{ann}_A(fA)$ is homogeneous. □

Scholium III.3.2. *For any cyclic submodule of $\mathcal{M}$ generated by an element of weak minimal length, there is a cyclic homogeneous submodule of $\mathcal{M}$ with the same annihilator.*

Proof of Theorem III.1.1: Given a prime submodule $\mathcal{N} \leq \mathcal{M}$, select $f \in \mathcal{N}$ of minimal length. Since $\mathcal{N}$ is prime, $\text{ann}_A(fA) = \text{ann}_A(\mathcal{N})$ is homogeneous, by Corollary II.3.1. Without loss of generality, we may write $f = \sum_{i=0}^{t} mu^{a_i}$, where $0 \leq a_0 < \cdots < a_r = 0$. Let $\alpha = m_0 u^{a_0}$.

We claim that αA is a prime submodule of $\mathcal{M}$ with $\text{ann}_A(\mathcal{N}) = \text{ann}_A(\alpha A)$. Since α is homogeneous, $\text{ann}_A(\alpha A)$ is homogeneous, and by the proof of Lemma III.3.1, $\text{ann}_A(\alpha A) = \text{ann}_A(fA) = \text{ann}_A(\mathcal{N})$. It remains to show that αA is prime.

Suppose αA is not prime. Then there exists a submodule of αA whose annihilator strictly contains $\mathcal{N}_A(\alpha A)$. Therefore we may choose an element $h \in \alpha A$ of weak minimal length such that $\text{ann}_A(hA)$ strictly contains $\text{ann}_A(\alpha A)$. Moreover, we may write $h = \alpha g$ for some $g \in A$, and further assume that no term of g annihilates α. By III.3.1, $\text{ann}_A(hA)$ is homogeneous. Choose a homogenous element $s \in \text{ann}_A(hA)$. Since $hAs = 0$, it follows that $\alpha g R u^i s = 0$ and $\alpha g R d^i s = 0$ for all $i \geq 0$. As $h = \alpha g$ is of weak minimal length, we observe that if γ is the term of least degree of g, then $\alpha\gamma R u^i s = 0$ and $\alpha\gamma R d^i s = 0$ for all $i \geq 0$. By minimal length, $f\gamma R u^i s = 0$ and

$f\gamma R d^i s = 0$ for all $i \geq 0$. Thus, $s \in \operatorname{ann}_A(f\gamma A)$. But $f\gamma \in fA$, and fA is prime. Therefore, $s \in \operatorname{ann}_A(fA) = \operatorname{ann}_A(\alpha A)$, a contradiction. Thus, no such element exists, and consequently, αA is prime. □

By Theorem III.1.1, it suffices to consider homogeneous prime submodules of $\mathcal{M}$. We can easily reduce further to the case of cyclic homogeneous prime submodules. Following the discussion in Section III.1, we see that there are three possible cases: there are strictly positive prime submodules, strictly negative prime submodules, and prime submodules which are manifest in all degrees. There is an easy way to separate the strictly positive and negative cases from the last case, as in the next lemma.

Lemma III.3.3. *$\mathcal{M}$ has has a strictly positive [strictly negative] submodules if and only if M has a nonzero $\sigma^i(q)$-torsion submodule for some $i > 0$ [$i \leq 0$].*

Proof: Suppose $\mathcal{L} \leq \mathcal{M}$ is a strictly positive submodule. Choose $f \in \mathcal{L}$ of weak minimal length, such that the homogeneous term of least degree of f, say mu^a, is of minimal degree, a. Then $fRd = 0$, as any nonzero element would contradict the minimality conditions on f. Consequently, $mu^a Rd = 0$. That is, $mR\sigma^a(q) = 0$. Thus, mR is a $\sigma^a(q)$-torsion submodule of M. Conversely, if $0 \neq N \leq M$ is $\sigma^i(q)$-torsion for some i, then the submodule of $\mathcal{M}$ generated by $N \otimes u^i$ is strictly positive. To see this we observe that $\sigma^i(q)$ is factor of the coefficent of $u^{i+k}d^j$ for all $k \geq 0$ and $j \geq k$. That is, $N \otimes u^{i+k}d^j = 0$ for all such k and j.

The strictly negative case is treated similarly. □

Our first goal is to describe $\operatorname{ann}_R(\alpha A)$, where α is homogenous and αA is a prime A-submodule of $\mathcal{M}$. We must separate the cases when αA is strictly positive or strictly negative from the case when $\alpha \in M$.

Proposition III.3.4. *If $m \in M$ and mA is prime, then $\operatorname{ann}_R(mA) = (\operatorname{ann}_R(mR))_{\sigma^*}$ and mR is Laurent σ-prime.*

Proof: Let $I = (\operatorname{ann}(mR))_{\sigma^*}$. We know that $mAr = 0$ if and only if $mRu^i r = 0$ and $mRd^i r = 0$ for all $i \geq 0$. That is, if and only if $mR\sigma^i(r) = 0$ for all $i \in \mathbb{Z}$. Equivalently, $r \in \sigma^i(\operatorname{ann}_R(m_0 R))$ for all such i, which by definition, means $r \in I$.

It remains to show that mR is Laurent σ-prime. Let $n \in mR$. Then there exists $r \in R$ such that $mr = n$. To see $(\operatorname{ann}(nR))_{\sigma^*} = I$, we apply a similar argument as the one above to the element mr. We conclude that $\operatorname{ann}_R(mrA) = (\operatorname{ann}_R(mrR))_{\sigma^*}$. Let $I' = \operatorname{ann}_R(mrA)$. If I' strictly contains I, then it must be that $\operatorname{ann}_A(mrA)$ strictly contains $\operatorname{ann}_A(mA)$, a contradiction. Thus $I' = I$, and so mR is Laurent σ-prime. $\square$

Lemma III.3.5. *Suppose L is a right R-module. Then L is σ-prime if and only if L^σ is σ-prime.*

Proof: Suppose L^σ is σ-prime. First, we note that $(\operatorname{ann}(L^\sigma))_\sigma = \bigcap_{i \in \mathbb{N}} \sigma^{-i}(\operatorname{ann}(L^\sigma))$ is a σ-invariant ideal. Since σ is surjective, $(\operatorname{ann}(L))_\sigma =$

$$\bigcap_{i \in \mathbb{N}} \sigma^{-i}(\operatorname{ann}(L)) = \sigma\left(\bigcap_{i \in \mathbb{N}} \sigma^{-i-1}(\operatorname{ann}(L^\sigma)\right) = \sigma((\operatorname{ann}(L^\sigma))_\sigma) = (\operatorname{ann}(L^\sigma))_\sigma.$$

If $0 \neq K < L$, then $I = (\text{ann}(K))_\sigma$ contains $J = (\text{ann}(L))_\sigma = (\text{ann}(L^\sigma))_\sigma$. Now $\sigma^{-1}(I) = \text{ann}(K^\sigma)_\sigma$. Since L^σ is σ-prime, $\sigma^{-1}(I) = J$. Again, σ is onto so $\sigma((\sigma^{-1}(I))) = \sigma(J) = J$. But $\sigma((\sigma^{-1}(I))) = I$, and thus $I = J$. Therefore L is also σ-prime.

Conversely, if L is σ-prime, and $0 \neq K \leq L^\sigma$, then since σ is an automorphism, $N = K^{\sigma^{-1}}$ is a submodule of L with $N^\sigma = K \leq L^\sigma$. It is then easy to see that $(\text{ann}(L^\sigma)) = \sigma^{-1}((\text{ann}(L))_\sigma) = (\text{ann}(L))_\sigma$, and $(\text{ann}(K)) = \sigma^{-1}((\text{ann}(N))_\sigma) = (\text{ann}(N))_\sigma = (\text{ann}(L))_\sigma$. □

Proposition III.3.6. *Suppose that $\alpha = mu^a \in \mathcal{M}$ is homogeneous, $a > 0$, $\alpha A \leq \mathcal{M}_A$ is prime, and no homogeneous element of αA has degree less than a. Then $\text{ann}_R(\alpha A) = (\text{ann}(mR))_\sigma$, and mR is σ-prime.*

Proof: Let $I = (\text{ann}(mR^{\sigma^a}))_\sigma$. For $r \in R$, $\alpha A r = 0$ if and only if $\alpha R u^i r = 0$ for all $i \geq 0$. Preceeding as in the proof of Proposition III.3.4, this holds for r if and only if $mu^a R u^i r = mu^a R \sigma^i(r) u^i = mR\sigma^{a+i}(r) u^{a+i} = 0$ for all $i \geq 0$. That is, if and only if, $r \in \sigma^{-i}(\text{ann}(mR^{\sigma^a}))$ for all $i \geq 0$, which by definition is exactly when $r \in I$. A similar argument where m is replaced by an arbitrary $n \in mR$ shows that $I = (\text{ann}((nR)^{\sigma^a}))_\sigma$ for all $n \in mR$. To see that I is σ-invariant, we consider $\text{ann}_R(\alpha u A)$. We deduce using the same reasoning as above, that $\text{ann}_R(\alpha u A) = \sigma^{-1}(I)$. Since αA is a prime A-module, we conclude that $\sigma^{-1}(I) = I$. Therefore $(mR)^{\sigma^a}$ is σ-prime. Since $a > 0$, the preceeding lemma shows that $mR^{\sigma^{a-1}}, \ldots, mR^\sigma, mR$ are also σ-prime, and since σ is an automorphism and I is σ-invariant, $(\text{ann}(mR))_\sigma = \sigma^a(I) = I$. □

It is clear that the analogous result with $a < 0$ and σ^{-1} in place of σ holds.

Proof of Theorem III.1.2: If the prime module $\mathcal{N}_A$ has a strictly positive or strictly negative submodule, then we can replace $\mathcal{N}$ by a cyclic homogenous strictly positive or strictly negative submodule without changing $J = \mathrm{ann}_A(\mathcal{N})$. By the above, $J \cap R = \mathrm{ann}_R(\mathcal{N})$ is either a σ- or σ^{-1}-associated ideal. We know that any σ- or σ^{-1}-associated ideal is also a Laurent σ-associated ideal.

If $\mathcal{N}$ has no strictly positive or strictly negative submodules, we can replace $\mathcal{N}$ by a homogeneous prime ideal of the form mA for some $m \in M$. Proposition III.3.4 shows that $\mathrm{ann}_A(\mathcal{N}) \cap R$ is a Laurent σ-associated ideal directly. □

Given a cyclic homogeneous prime submodule αA, we now characterize $\mathrm{ann}_A(\alpha A)$. In general, there is a split between those prime modules which are strictly positive or strictly negative, and those modules which are manifest in all degrees. It is tempting to think that the dividing factor is between those modules having either d^i- or u^i-torsion, for some $i > 0$, and those that do not. Although this is also the case, the main the fracture takes place along the relative size of the submodule. The latter case regarding d^i- and u^i-torsion is easy to grasp, as shown in the following result.

Proposition III.3.7. *If $\alpha A \leq \mathcal{M}$ is prime and strictly positive, then $d^j \in \mathrm{ann}_A(\alpha A)$ if and only if $(j, -j) \in \mathrm{ann}(\alpha A)$. If αA is strictly negative, then $u^j \in \mathrm{ann}(\alpha A)$ if and only if $(j, -j) \in \mathrm{ann}(fA)$.*

Proof: Again, we prove the result when αA is strictly positive, the strictly negative case is similar. We assume that α is minimal in the sense that no homogeneous element of αA has degree less than that of α. Since $\mathrm{ann}_A(\alpha A)$ is a two-sided ideal of A, if $d^j \in \text{ann}(\alpha A)$, then $u^j d^j = \sigma^j(q) \cdots \sigma(q) \in \text{ann}(\alpha A)$. Conversely, suppose that $\sigma^{j-1}(q) \cdots \sigma(q) \in \text{ann}(\alpha A)$. We know that $\alpha u^j \in \alpha A$, and since αA is strictly positive, $\alpha A = \alpha R + \alpha R u + \alpha R u^2 + \cdots$. Consequently, $\alpha u^j A = \alpha A u^j$. Since αA is prime, $\text{ann}(\alpha u^j A) = \text{ann}(\alpha A)$. Thus, $\alpha u^j A d^j = \alpha A u^j d^j = \alpha A \sigma^j(q) \cdots \sigma(q) = 0$. Hence $d^j \in \text{ann}(\alpha u^j A) = \text{ann}(\alpha A)$. □

Corollary III.3.8. *If αA is a strictly positive [strictly negative] prime module and $gw(ann_R(\alpha A)) = 0$, then d^j [u^j] is not in $\text{ann}_A(\alpha A)$.*

Corollary III.3.9. *If R is Noetherian or σ has finite order, then every strictly positive [strictly negative] prime submodule $\mathcal{N}$ of $\mathcal{M}$ is d^k-torsion [u^k-torsion], where $k = gw(\text{ann}_R(\mathcal{N}))$.*

Proof: Any such submodule corresponds to a σ-prime [σ^{-1}-prime] submodule N of M_R which has $\sigma^i(q)$-torsion for some $i > 0$ [$i \leq 0$] by III.1.1 and III.3.3. Consequently, $(k, -k) \in (\text{ann}(N))_{\sigma^*} = (\text{ann}(M))_{\sigma^*} = \text{ann}_R(\mathcal{N})$ where $k = gw((\text{ann}(M))_{\sigma^*})$. Thus, Proposition III.3.7 says that $\mathcal{N}$ is thus d^k-torsion [u^k-torsion]. □

As seen in Example III.2.4, one can easily construct a module over a non-Noetherian ring in which there are strictly positive prime submodules of the induced module which are not annihilated by u^j for any $j > 0$. We have just seen this is not the case when working over a Noetherian ring, or when σ has finite order.

Proposition III.3.10. *If* $m \in M$ *and* $I = (\text{ann}(mR))_{\sigma^*}$, *then*

1. $\text{ann}_R(mA) = I$;

2. $\text{ann}_A(mA) = \sum_{k \in \mathbb{Z}} I_k$, *where*

$$I_k = \begin{cases} ((k,-k) : I) \cap (\text{ann}(mR))_\sigma & \textit{for} \;\; k > 0 \\ I & \textit{for} \;\; i = 0 \\ ((k,-k) : I) \cap (\text{ann}(mR))_{\sigma^{-1}} & \textit{for} \;\; k < 0. \end{cases}$$

Proof: The first statement is obvious, as $mAr = 0$ if and only if $mRu^i r = mR\sigma^i(r)u^i = 0$ and $mRd^i r = mR\sigma^{-i}(r)d^i = 0$ for all $i \geq 0$. To prove the second we show that $ru^k \in \text{ann}(mA)$ if and only if $r \in ((k,-k) : I) \cap (\text{ann}(mR))_\sigma$. For $k \geq 0$, set $I_k = \{r \in R \mid ru^k \in \text{ann}(fA)\}$. Observe that $mRu^j ru^k = 0$ exactly when $r \in (\text{ann}(mR))_\sigma$, so this is a necessary condition on $r \in I_k$. To see that $I_k = ((k,-) : I) \cap (\text{ann}(mR))_\sigma$, we proceed by induction on $k \geq 0$. The case $k = 0$ is clear, since $I_0 = ((0,0) : I) \cap (\text{ann}(mR))_\sigma = I \cap (\text{ann}(mR))_\sigma = I$. So suppose for some $l > 0$, that for all $0 \leq k < l$, $I_k = ((k,-k) : I) \cap (\text{ann}(mR))_\sigma$. Consider $r \in I_l$. It is clear that $r \in (\text{ann}(mR))_\sigma$. To see why $r \in ((k,-k) : I)$, we observe that since $\text{ann}(mA)$ is an ideal of A, and $ru^l \in \text{ann}(mA)$, $ru^l d = r\sigma^l(q)u^{l-1} \in \text{ann}(mA)$. By the induction hypothesis $r\sigma^l(q) \in I_{l-1} = ((l-1, 1-l) : I)$. Thus $r\sigma^l(q)(l-1, 1-l) = r\sigma^l(q)\sigma^{l-1} \cdots \sigma(q) = r(l,-l) \in I$. Therefore $I_l \subseteq ((l,-l) : I) \cap (\text{ann}(mR))_\sigma$.

On the other hand, if $r \in ((l,-l) : I) \cap (\mathrm{ann}(mR))_\sigma$, then we need to show $mAru^l = 0$. Since $r \in (\mathrm{ann}(mR))_\sigma$, $mRu^jru^l = 0$ for all $j \geq 0$. Next, $mRd^jru^l = mR\sigma^{-j}(r)d^ju^l$. When $j = 0$, $mRru^l = 0$, as $r \in (\mathrm{ann}(mR))_\sigma$. For $j > 0$, $mRd^jru^l = mRd^{l-1}\sigma^{-1}(r)qu^{l-1}$. Thus it will suffice to show, by our induction hypothesis, that $\sigma^{-1}(r)q \in I_{l-1}$. Well, $\sigma^{-1}(r)q(l-1, 1-l) = \sigma^{-1}(r)q\sigma^{l-1}(q)\ldots\sigma(q) = \sigma^{-1}(r)\sigma^{l-1}(q)\cdots q$. By our initial supposition, $\sigma(\sigma^{-1}(r)\sigma^{l-1}(q)\cdots q) = r(l,-l) \in I$. Since I is σ-invariant, $\sigma^{-1}(r)\sigma^{l-1}(q)\cdots q \in I$, and thus $\sigma^{-1}(r)q \in I_{l-1}$, as desired. Therefore $((l,-l) : I) \cap (\mathrm{ann}(mR))_\sigma \subseteq I_l$. We conclude that $I_k = ((k,-k) : I) \cap (\mathrm{ann}(mR))_\sigma$ for all $k \geq 0$.

The case of $rd^k \in \mathrm{ann}(mA)$ is treated similarly, and we see that $I_k = ((k,-k) : I) \cap (\mathrm{ann}(mR))_{\sigma^{-1}}$ for $k \leq 0$. Therefore the annihilator of mA is as stated. □

Lemma III.3.11. *Suppose N_R is Laurent σ-prime and $\sigma^i(q)$-torsion-free for all $i \in \mathbb{Z}$. If $I = (\mathrm{ann}_R(N))_{\sigma^*}$, then $((k,-k,) : I) = I$.*

Proof:. Observe that, since $(k,-k)$ is central and N is $\sigma^i(q)$-torsion-free for all $i \in \mathbb{Z}$, $N(k,-k)$ is a *nonzero* submodule of N_R for all $k \in \mathbb{Z}$. As N is Laurent σ-prime, $(\mathrm{ann}_R(N(k,-k)))_{\sigma^*} = ((k,-k) : \mathrm{ann}_R(N))_{\sigma^*} = I$. We observe that $I \subseteq ((k,-k) : I) \subseteq ((k,-k) : \mathrm{ann}(N))_{\sigma^*}$. Consequently $I = ((k,-k) : I)$. □

Corollary III.3.12. *If mA is prime and $I = (\mathrm{ann}(mR))_\sigma$, then $\mathrm{ann}_A(mA) = IA$.*

The way we have written I_k in III.3.10 shows that $(\text{ann}(mR))_{\sigma^*} \subseteq I_k \subseteq (\text{ann}(mR))_\sigma$ for $k > 0$ and $(\text{ann}(mR))_{\sigma^*} \subseteq I_k \subseteq (\text{ann}(mR))_{\sigma^{-1}}$ for $k < 0$. From this we deduce the following result regarding prime submodules of $\mathcal{M}$ when working either over a Noetherian ring, or with an automorphism of finite order, since, in that case $(\text{ann}(mR))_{\sigma^*} = (\text{ann}(mR))_\sigma = (\text{ann}(mR))_{\sigma^{-1}}$. The result also follows directly from III.3.12.

Corollary III.3.13. *If R is Noetherian or σ has finite order, $mA \leq \mathcal{M}_A$ is prime and $I = \text{ann}_R(mA)$, then $\text{ann}_A(mA) = IA$.*

In these cases we can also build prime submodules of $\mathcal{M}_A$ which are manifest in all degrees from Laurent σ-prime submodules of M.

Corollary III.3.14. *If M is Laurent σ-prime and $\sigma^i(q)$-torsion-free for all $i \in \mathbb{Z}$, and $I = (\text{ann}_R(M))_{\sigma^*}$, then $\mathcal{M} = M \otimes_R A$ is prime with annihilator IA.*

Proof: Let $I = (\text{ann}(M))_{\sigma^*}$. For any $0 \neq m \in M$, mR is also σ-prime. So it suffices to show that $mR \otimes A$ is prime and that $\text{ann}(mR \otimes A)$ is constant over all $m \in M$.

To see that $mR \otimes A$ is prime, we note that it is generated by the homogeneous polynomial m [e.g. $m \otimes 1$], which is naturally of weak minimal length. By III.3.10 and III.3.11, $\text{ann}(mR \otimes A) = IA$. Since $\{\sigma^i(q)\}_{i \in \mathbb{Z}}$ is disjoint from all annihilators of nonzero submodules of M, it follows that every nonzero submodule $\mathcal{N} \leq \mathcal{M}$ has an element, g, of weak minimal length which has a nonzero homogenous term of degree zero. Without loss of generality, $g = \sum_{i=0}^{t} m_i u^{a_i}$ and $a_0 = 0$. Let $n = m_0$. Observe that $\operatorname{ann}(gA) \supseteq \operatorname{ann}(\mathcal{N}) \supseteq \operatorname{ann}(mR \otimes A)$. Now $n \in mR$, and so $\operatorname{ann}(gA_R) = (\operatorname{ann}(nR))_{\sigma^*} = (\operatorname{ann}(mR))_{\sigma^*} = \operatorname{ann}((mR \otimes A)_R) = I$, as mR is Laurent σ-prime. Thus, by Lemma III.3.11 and Proposition III.3.10, $\operatorname{ann}_A(gA) = IA$. Thus $\operatorname{ann}(gA) = \operatorname{ann}(mR \otimes A)$. A similar argument shows that $\operatorname{ann}_A(m'R \otimes A) = \operatorname{ann}_A(mR \otimes A) - IA$ for any nonzero $m' \in M$. Therefore $M \otimes_R A$ is prime with annihilator IA. □

Corollary III.3.15. *Suppose M_R is Laurent σ-prime and $\sigma^i(q)$-torsion-free for all $i \in \mathbb{Z}$. Let $I = (\operatorname{ann}(M))_{\sigma^*}$. If σ has finite order or R is Noetherian, then $\mathcal{M} = M \otimes_R A$ is prime, with annihilator IA.*

We now turn to the case of computing the annihilators of strictly positive and strictly negative prime modules. We will show how to compute those of strictly positive modules; those of strictly negative modules can easily be deduced. We begin with a more general proposition regarding cyclic modules generated by elements of weak minimal length.

Proposition III.3.16. *Consider a strictly positive cyclic A-submodule of the form mu^aA, where a is the minimum degree among elements of mu^aA. If $I = (\operatorname{ann}_R((mR))_\sigma$ is σ-invariant, then* $\operatorname{ann}_A(mu^aA) = \sum_{k \le 0}((k,-k) : I)d^{-k} + \sum_{k>0} Iu^k$.

Proof: It is clear, since u^k acts without torsion on any strictly positive module, that $ru^k \in \operatorname{ann}(mu^aA)$ if and only if $r \in \operatorname{ann}(mu^aA_R)$ for $k \ge 0$. By Proposition III.3.6, $\operatorname{ann}(mu^aA_R) = (\operatorname{ann}(mR))_\sigma = I$. To show that $rd^k \in \operatorname{ann}(mu^aA)$ if and only if $r \in ((-k,k) : I)$, we proceed by induction on k. For $k = 0$, we note that $rd^k = r$, and $r \in \operatorname{ann}(mu^a A)$ if and only if $r \in I = (1 : I) = ((0,0) : I)$. Now assume that for some $l > 0$, for all $0 \leq k < l$, $rd^k \in \operatorname{ann}(mu^a A)$ if and only if $r \in ((-k,k)) : I)$. Let $J_l = \{r \in R \mid rd^l \in \operatorname{ann}(mu^a A)\}$. We show $J_l = ((-l,l) : I)$. First, if $rd^l \in \operatorname{ann}(mu^a A)$, then $urd^l = \sigma(r)\sigma(q)d^{l-1} \in \operatorname{ann}(mu^a A)$, as $\operatorname{ann}(mu^a A)$ is a two-sided ideal of A. By the induction hypothesis, $\sigma(r)\sigma(q) \in ((1-l, l-1) : I)$. That is, $\sigma(l)\sigma(q)(\sigma^{2-l}(q)\cdots\sigma^{-1}(q)q) = \sigma(r)\sigma(q)q\sigma^{-1}(q)\cdots\sigma^{2-l}(q) \in I$. Since I is σ-invariant, we may apply σ^{-1}, and so $rq\sigma^{-1}(q)\cdots\sigma^{1-l}(q) = r(-l,l) = r(-l,l) \in I$. Equivalently $r \in ((-l,l) : I)$. Therefore, $J_l \subseteq ((-l,l) : I)$.

Now suppose $r \in ((-l,l) : I)$. We show $mRu^{a+j}rd^l = 0$, for all $j \geq 0$. Observe that $mRu^{a+j}rd^l = mR\sigma^{a+j}(q)u^{a+j-1}\sigma^{-1}(r)d^{l-1}$. Since a is the minimal degree of elements of $mu^a A$, $mR\sigma^a(q) = 0$. Thus, for $j = 0$, $mRu^{a+j}rd^l = mR\sigma^a(q)\sigma^a(r)u^{a-1}d^{l-1} = 0$. For $j > 0$, we need to show $mR\sigma^{a+j}(q)u^{a+j-1}\sigma^{-1}(r)d^{l-1} = 0$. Reindexing with $k = j-1$, $mRu^{a+j-1}rd^l = mRu^{a+k}\sigma(r)\sigma(q)d^{l-1} = 0$ for all $k \geq 0$ if and only if $\sigma(r)\sigma(q) \in ((1-l, l-1) : I)$, by the induction hypothesis. Well, if $r \in ((-l,l) : I)$, then $\sigma(r)\sigma(q)(1-l, l-1)) = \sigma(r)\sigma(q)q\sigma^{-1}(q)\cdots\sigma^{2-l}(q)$. Now

$$\sigma^{-1}(\sigma(r)\sigma(q)q\sigma^{-1}(q)\cdots\sigma^{2-l}(q)) = rq\cdots\sigma^{1-l}(q) = r(-l,l) \in I,$$

as $r \in ((-l,l) : I)$. Since I is σ-invariant, $\sigma(r)\,\sigma(q)\,q\,\sigma^{-1}(q)\,\cdots\,\sigma^{2-l}(q) \in I$, whence

$\sigma(r)\sigma(q) \in ((1-l, l-1) : I)$. Therefore $mRu^{a+j}rd^l = 0$ for all $j \geq 0$, and so $r \in J_l$. Thus, $J_l = ((-l, l) : I)$. We conclude that

$$\mathrm{ann}(mu^a A_A) = \sum_{k \leq 0} ((k, -k) : I)d^{-k} + \sum_{k>0} Iu^k. \ \square$$

The analogous result for strictly negative modules, below, also holds. The proof is similar and is omitted.

Scholium III.3.17. *Suppose md^aA is strictly negative, where a is the maximum degree of elements of md^aA. If $I = (\mathrm{ann}(mR))_{\sigma^{-1}}$ is σ^{-1}-invariant, then* $\mathrm{ann}(md^aA_A) = \sum_{k<0} Id^{-k} + \sum_{k \geq 0} ((k, -k) : I)u^k$.

Corollary III.3.18. *If αA is a prime strictly positive [strictly negative] homogeneous submodule of $\mathcal{M}$ and $I = \mathrm{ann}_A(\alpha A) \cap R$, then* $\mathrm{ann}_A(\alpha A) = \sum_{k \leq 0} (\sigma^{-k}((k, -k)) : I)d^{-k} + \sum_{k>0} Iu^k$ *[$\sum_{k<0} Id^{-k} + \sum_{k \geq 0} ((k, -k)) : I)u^k$].*

This shows for every strictly positive, or strictly negative, prime submodule $\mathcal{N} \leq \mathcal{M}_A$, $\mathrm{ann}_A(\mathcal{N})$ can be built from $\mathrm{ann}_R(\mathcal{N})$. Conversely, we can build homogeneous strictly positive and strictly negative prime submodules out of certain σ- and σ^{-1}-prime submodules of M, respectively.

Corollary III.3.19. *Let M be a right R-module.*

1. *If M is σ-prime, $I = (\mathrm{ann}(M))_\sigma$ and $\sigma^i(q) \in \mathrm{ann}(M)$ for some $i > 0$, then $\mathcal{N} = \sum_{j \geq i} M \otimes u^j \leq M \otimes_R A$ is prime with*

$$\mathrm{ann}_A(\mathcal{N}) = \sum_{k \leq 0} ((k, -k) : I)d^{-k} + \sum_{k>0} Iu^k.$$

2. *If M is σ^{-1}-prime, $I = (\mathrm{ann}(M))_{\sigma^{-1}}$ and $\sigma^i(q) \in \mathrm{ann}(N)$ for some $i \leq 0$, then $\mathcal{N} = \sum_{j > -i} M \otimes d^j \leq M \otimes_R A$ is prime with*

$$\mathrm{ann}_A(\mathcal{N}) = \sum_{k<0} I d^k + \sum_{k \geq 0} ((k, -k) : I) u^{-k}.$$

Proof: We prove the first statement, the second is proved similarly. To see that $\mathcal{N}$ is indeed a submodule of $N \otimes_R A$, observe that $\sigma^i(q)$ is always a factor of $(j, -k)$ for all $k \geq j \geq i$. Consequently, $\mathcal{N}$ is closed under the action of d^k, and is clearly closed under the action of R and u^k for all $k > 0$. Given $m \in N$, the homogeneous polynomial mu^i satisfies the hypotheses of Proposition III.3.16, since $(\mathrm{ann}((mR)^{\sigma^i}))_\sigma = \sigma^{-i}((\mathrm{ann}(mR))_\sigma) = I$, as M is σ-prime. Thus $\mathrm{ann}(mu^iA) = \sum_{k \leq 0} ((k, -k) : I) d^{-k} + \sum_{k>0} I u^k$. Clearly $\mathrm{ann}_A(\mathcal{N})$ is the intersection of the annihilators of mu^iA, as m ranges over all nonzero element of M. Since M is σ-prime, $\mathrm{ann}_A(mu^iA)$ is constant over that range. Hence $\mathrm{ann}(\mathcal{N}) = \sum_{k \leq 0} ((k, -k) : I) d^{-k} + \sum_{k>0} I u^k$.

To see that $\mathcal{N}$ is prime, let $0 \neq \mathcal{L} \leq \mathcal{N}$. Choose an element, $0 \neq g = \sum_{i=0}^{t} n_i u^{b_i} \in \mathcal{L}$ of weak minimal length, so that b_0 is minimal among all elements of weak minimal length in $\mathcal{L}$. As we have seen before, since $n_0 \in N$,

$$\mathrm{ann}(gA_R) = (\mathrm{ann}(n_0 R^{\sigma^{b_0}}))_\sigma = \sigma^{-b_0}((\mathrm{ann}(n_0 R))_\sigma) = I.$$

So $\mathrm{ann}(gA) = \mathrm{ann}(\mathcal{N}) = \sum_{k \leq 0} ((k, -k) : I) d^{-k} + \sum_{k>0} I u^k$. But $\mathrm{ann}(gA) \supseteq \mathrm{ann}(\mathcal{L}) \supseteq \mathrm{ann}(\mathcal{N})$. Therefore $\mathrm{ann}(\mathcal{L}) = \mathrm{ann}(\mathcal{N})$. □

Corollary III.3.20. *Suppose R is Noetherian or σ has finite order. If $N \leq M$ is σ^*-prime, $I = (\text{ann}(N))_{\sigma^*}$, and $k = gw(I) > 0$, then there is a strictly positive prime submodule of $M \otimes_R A$ which is d^k-torsion and a strictly negative prime submodule of $M \otimes_R A$ which is u^k-torsion.*

Proof: If R is Noetherian or σ is of finite order, then σ-associated, σ^{-1}-associated and Laurent σ-associated ideals coincide. Next we note that since $gw(I) > 0$, there exists a nonzero submodule, K, of N that is $\sigma(q)^s$-torsion for some $s > 0$. To see this, note at least one of $N, N\sigma^k(q), N\sigma^k(q)\sigma^{k-1}(q), \dots N\sigma^k(q)\cdots\sigma^2(q)$ is nonzero and annihilated by σ^s, for some $1 \leq s \leq k$. Similarly, for some $1-k \leq t \leq 0$, there is nonzero $\sigma^t(q)$-torsion submodule, L of N. Then $\sum_{l \geq s} K \otimes u^l$ and $\sum_{l > -t} K \otimes d^l$ are prime is clear by the preceding result. Indeed, since $(k, -k) \in I$, it follows that these submodules are d^k-torsion and u^k-torsion, respectively. □

Proof of Theorem III.1.4: Let $J \in \text{Ass}(\mathcal{M}_A)$. If J is the annihilator of a strictly positive or strictly negative prime submodule of $\mathcal{M}$, then J can be built from $I = J \cap R$ via the formula given in III.3.16 or III.3.17. Otherwise, J is the annihilator of a prime submodule which has no strictly positive or strictly negative submodules. By Proposition III.3.4, there exists a Laurent σ-prime submodule $N \leq M$ whose Laurent σ-associated ideal is $J \cap R$. Furthermore, III.3.10 states that J an be constructed from $J \cap R$ and the ideals $(\text{ann}(N))_\sigma$ and $(\text{ann}(N))_{\sigma^{-1}}$ using the prescribed formula.

□

Proof of Corollaries III.1.5 and III.1.6: Let $\mathcal{S} = \{\sum_{k\leq 0}((k,-k) : I)d^{-k} + \sum_{k>0} Iu^k \mid I \in \sigma\text{-}\operatorname{Ass}(M),\ gw(I) > 0\}$

$$\cup \{\sum_{k<0} Id^k + \sum_{k\geq 0}((k,-k) : I)u^{-k} \mid I \in \sigma\text{-}\operatorname{Ass}(M),\ gw(I) > 0\}$$
$$\cup \{IA \mid I \in \sigma\text{-}\operatorname{Ass}(M),\ gw(I) = 0\}.$$

Let $J \in \operatorname{Ass}(\mathcal{M})$ and $I = J \cap R$. Then J can be constructed from I, as in III.1.4. When R is Noetherian or σ is finite order, we have already seen that σ-, σ^{-1}- and Laurent σ-associated ideals coincide. In the Noetherian and finite order cases, each of the constructions in the three statements of III.1.4 yields a distinct ideal. To see this, we observe the following. If $gw(I) > 0$, then J contains d^k or u^k for some $k > 0$, by III.3.9. Thus $J = \sum_{k\leq 0}((k,-k) : I)d^{-k} + \sum_{k>0} Iu^k$ or $J = \sum_{k<0} Id^k + \sum_{k\geq 0}((k,-k) : I)u^{-k}$, respectively. If $gw(I) = 0$, then neither d^k nor u^k are in J for all $k > 0$. Consequently J must be the annihilator of a prime submodule of $\mathcal{M}_A$ of the form mA, for some $m \in M$ where mR is $\sigma^i(q)$-torsion-free for all $i \in \mathbb{Z}$. So $J = IA$, by III.3.13. Therefore $J \in \mathcal{S}$, and we conclude $\operatorname{Ass}(\mathcal{M}_A) \subseteq \mathcal{S}$.

Conversely, let $I \in \sigma^*\text{-}\operatorname{Ass}(M)$. If $gw(I) = 0$, then the proof of III.3.20 and III.3.7 show that any Laurent σ-prime submodule $N \leq M$ must be $\sigma^i(q)$-torsion-free for all $i \in \mathbb{Z}$. Consequently, III.3.15 shows that IA is an associated prime of $\mathcal{M}_A$. If $gw(I) > 0$, III.3.20, III.3.16, and III.3.17, show that I is in the image of Φ. Thus $\operatorname{Ass}(\mathcal{M}_A) \supseteq \mathcal{S}$. We conclude the two sets are equal and, moreover, that Φ is surjective. Additionally, III.3.15 implies that the A-annihilators of prime modules which are manifest in all degrees are of the form IA, where I is the R-annihilator. Thus Φ is one-to-one over Laurent σ-associated ideals with zero gw-index. If $I \in \sigma^*\text{-}\operatorname{Ass}(M)$ has $gw(I) > 0$, then III.3.20 says that I has exactly two preimages under Φ. That is, ϕ is two-to-one over Laurent σ-associated ideals with positive gw-index. $\square$

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