DIMENSION FUNCTIONS OF RATIONALLY
DILATED WAVELETS

by
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In *A characterization of dimension functions of wavelets*, Bownik, Rzeszotnik, and Speegle define and characterize the dimension function of an integer dilated wavelet in $L^2(\mathbb{R}^N)$. Furthermore, in *The wavelet dimension function for real dilations and dilations admitting non-MSF wavelets*, Bownik and Speegle define the dimension function for a rationally (in fact, real) dilated wavelet in $L^2(\mathbb{R})$. In this dissertation, we extend these two previous works to their logical intersection by defining the dimension function of a rationally dilated wavelet in $L^2(\mathbb{R}^N)$. We show that the definition previously given for integer dilated wavelets extends to include rationally dilated wavelets as well. We also show how the necessary (and sufficient) conditions for the dimension function of an integer dilated wavelet should be written to obtain necessary conditions for the dimension function of a rationally dilated wavelet.

Furthermore, we produce and characterize all wavelet sets in $\mathbb{R}$ consisting of exactly two or three intervals, thus extending an example previously given by Speegle in Bownik's work *On a problem of Daubechies*. 
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DEDICATION

This work is in honor of my father, who worked in a factory his entire career and taught me the value of mathematics at a very young age. I thank him and I will forever be in his debt.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td>4</td>
</tr>
<tr>
<td>II.1</td>
<td>4</td>
</tr>
<tr>
<td>II.2</td>
<td>8</td>
</tr>
<tr>
<td>II.3</td>
<td>14</td>
</tr>
<tr>
<td>II.4</td>
<td>31</td>
</tr>
<tr>
<td>III</td>
<td>38</td>
</tr>
<tr>
<td>III.1</td>
<td>38</td>
</tr>
<tr>
<td>III.2</td>
<td>42</td>
</tr>
<tr>
<td>III.3</td>
<td>50</td>
</tr>
<tr>
<td>III.4</td>
<td>67</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>80</td>
</tr>
</tbody>
</table>

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# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>II.1 Dilation partition condition for two intervals.</td>
<td>16</td>
</tr>
<tr>
<td>II.2 Translation partition condition for two intervals.</td>
<td>16</td>
</tr>
<tr>
<td>II.3 Dilation partition condition for three intervals.</td>
<td>18</td>
</tr>
<tr>
<td>II.4 Translation partition condition for three intervals. (Option 1)</td>
<td>18</td>
</tr>
<tr>
<td>II.5 Translation partition condition for three intervals. (Option 2)</td>
<td>19</td>
</tr>
<tr>
<td>II.6 $\mathcal{F}_1(a,p)$. Feasible region defined by (II.14) &amp; (II.15).</td>
<td>22</td>
</tr>
<tr>
<td>II.7 $\mathcal{F}_1(a,1)$.</td>
<td>23</td>
</tr>
<tr>
<td>II.8 $\mathcal{F}_1(a,p)$ for $a &gt; 2$.</td>
<td>24</td>
</tr>
<tr>
<td>II.9 $\mathcal{F}_1(2,p)$.</td>
<td>25</td>
</tr>
<tr>
<td>II.10 $\mathcal{F}_2(a,p)$. Feasible region defined by (II.18) &amp; (II.19).</td>
<td>25</td>
</tr>
<tr>
<td>II.11 $\mathcal{F}_2(a,1)$.</td>
<td>26</td>
</tr>
<tr>
<td>III.1 Dimension function $\mathcal{D}_\phi$ for Example III.4.1</td>
<td>68</td>
</tr>
<tr>
<td>III.2 Applying Algorithm III.2.5 to Example III.4.1.</td>
<td>69</td>
</tr>
<tr>
<td>III.3 Dimension function $\mathcal{D}_\phi$ for Example III.4.1</td>
<td>73</td>
</tr>
<tr>
<td>III.4 Dimension function for Example III.4.5</td>
<td>77</td>
</tr>
</tbody>
</table>
CHAPTER I

PRELIMINARIES

We begin by making explicit some definitions and conventions that will be used throughout this work. For \( f \in L^1(\mathbb{R}^N) \) we define the Fourier transform of \( f \) to be:

\[
\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i (x, \xi)} \, dx.
\]

We then use Plancherel's Theorem, which states that for all \( f, g \in L^2(\mathbb{R}^N) \) we have the relation \( \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle \), along with the fact that \( L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \) is dense in \( L^2(\mathbb{R}^N) \) to extend the Fourier transform to \( L^2(\mathbb{R}^N) \). Furthermore, notice that Plancherel's Theorem can be restated as \( \int_{\mathbb{R}^N} f(x) \overline{g(x)} \, dx = \int_{\mathbb{R}^N} \hat{f}(\xi) \overline{\hat{g}(\xi)} \, d\xi \). In particular, this implies that \( \| f \| = \| \hat{f} \| \) for all \( f \in L^2(\mathbb{R}^N) \).

Regarding symbols, we use \( \mathbb{T}^N \) to denote the torus \( \mathbb{R}^N / \mathbb{Z}^N \), which is identified with \( (-\frac{1}{2}, \frac{1}{2}]^N \), \( 1_X \) will denote the characteristic function of the set \( X \), and the delimiters \( \cdot \cdot \cdot \) can be used to denote the modulus of a number, the order of a group, or the Lebesgue measure of a set, depending on context.

Occasionally we will make use of the greatest integer and least integer (also known as the floor and ceiling) functions. We will use the delimiters \( \lfloor \cdot \rfloor \) and \( \lceil \cdot \rceil \), respectively. That is, given any \( x \in \mathbb{R} \), \( \lfloor x \rfloor \) is the greatest integer less than or equal to \( x \) and \( \lceil x \rceil \) is the least integer greater than or equal to \( x \).
By a \textit{dilation}, we mean a linear transformation $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ all of whose eigenvalues exceed 1. We will make free use of the identification between linear transformations on $\mathbb{R}^N$ and their $N \times N$ matrix representations and write $A \in M_N(\mathbb{R})$. Furthermore, by \textit{integer dilation} or \textit{rational dilation} we mean specifically that $A \in M_N(\mathbb{Z})$ or $A \in M_N(\mathbb{Q})$, respectively.

By a \textit{lattice} we mean $P\mathbb{Z}^N$ for some invertible $P \in M_N(\mathbb{R})$. Furthermore, if $\Gamma = P\mathbb{Z}^N$ is a lattice and $\Gamma' = P'\mathbb{Z}^N$ a sublattice, we use the term \textit{transversal} to refer to a set of $|\det(P')/\det(P)|$ representatives of distinct cosets of $\Gamma'$ in $\Gamma$.

Given a family of functions $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^N)$ and a dilation $A$, the \textit{affine system} associated with $(\Psi, A)$ is the family of functions $\{\psi^\ell_{j,k} : j \in \mathbb{Z}; k \in \mathbb{Z}^N; \ell = 1, \ldots, L\}$ where $\psi^\ell_{j,k}$ is defined as:

$$\psi^\ell_{j,k}(x) = |\det(A)|^{\frac{1}{2}} \psi^\ell(A^j x - k).$$

In other words, $\psi^\ell_{j,k} = D_j T_k \psi^\ell$ where $D_j$ and $T_k$ are the dilation and translation operators on $L^2(\mathbb{R}^N)$ given by:

$$D_j f(x) = |\det(A)|^{\frac{1}{2}} f(A^j x)$$

$$T_k f(x) = f(x - k)$$

for all $f \in L^2(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$.

There are three basic properties of the dilation and translation operators which we will employ repeatedly in order to greatly reduce our computations. Thus, we take time to make them explicit. They are as follows: (i) $T_k^* = T_{-k}$, (ii) $\langle T_k f, T_k g \rangle = \langle f, g \rangle$, (iii) $T_k D_j = D_j T_{A^j k}$. The proof of each of these is elementary, as shown here:

\textbf{Proof.} Let $j \in \mathbb{Z}$, $k \in \mathbb{Z}^N$, and $f, g \in L^2(\mathbb{R}^N)$. The change of variables $x \mapsto x + k$ yields $\langle T_k f, g \rangle = \int_{\mathbb{R}^N} f(x - k)\overline{g(x)} dx = \int_{\mathbb{R}^N} f(x)\overline{g(x + k)} dx = \langle f, T_k g \rangle$, proving (i).
Furthermore, (ii) is simply a consequence of (i) by the argument
\[ (T_k f, T_k g) = (f, T^{-k} T_k g) = (f, g). \] Finally, for all \( x \in \mathbb{R}^N \) we have
\[ T_k D_j f(x) = |\det(A)|^{\frac{1}{2}} f(A^j(x - k)) = |\det(A)|^{\frac{1}{2}} f(A^j x - A^j k) = D_j T_{A^j k} f(x), \]
giving (iii).

\[\square\]

Of particular interest to us is a class of families of functions called wavelets. A \textit{wavelet} is any family of functions \( \Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^N) \) along with a dilation \( A \) such that the affine system associated with \( (\Psi, A) \) is an orthonormal basis of \( L^2(\mathbb{R}^N) \). Most often we employ the convention \( \Psi \) is a \textit{wavelet} to mean that \( \Psi \) is a wavelet when associated with some dilation \( A \). In this case, any important characteristics of the dilation will be made clear, as in the case of \( \Psi \) is an \textit{integer dilated wavelet} or \( \Psi \) is a \textit{rationally dilated wavelet} meaning that \( (\Psi, A) \) is a wavelet for some unspecified \( A \in M_N(\mathbb{Z}) \) or \( A \in M_N(\mathbb{Q}) \), respectively.

Finally, when \( \Psi = \{\psi^1, \ldots, \psi^L\} \) is a wavelet, we refer to the closed linear span of \( \{\psi^{\ell}_{j,k} : j < 0; k \in \mathbb{Z}^N; \ell = 1, \ldots, L\} \) as the space of negative dilates or the \textit{core space} of \( \Psi \) and denote it by \( V_0 \). That is:
\[ V_0 = \text{span} \{\psi^{\ell}_{j,k} : j < 0; k \in \mathbb{Z}^N; \ell = 1, \ldots, L\}. \]

In one dimension, wavelets consisting of a single function are sufficient for developing the entire "classical" (i.e. dyadic dilation) theory. Thus, in \( L^2(\mathbb{R}) \) we assume that \( L = 1 \) and we say \( \psi \in L^2(\mathbb{R}) \) is a wavelet if \( \{\psi_{j,k} : j, k \in \mathbb{Z}\} \) is an orthonormal basis of \( L^2(\mathbb{R}) \). Furthermore, the dilation \( A \in M_N(\mathbb{R}) \) with eigenvalues greater than 1 becomes \( a \in \mathbb{R} \) with \( a > 1 \). This is in contrast to dimensions greater than 1, where certain types of wavelets (namely, Multi-Resolution Analysis wavelets) necessitate the use of more than one function.
CHAPTER II

MINIMALLY SUPPORTED FREQUENCY WAVELETS

II.1 What's in a Name?

Suppose that $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^N)$ is a wavelet with the property that $|\hat{\psi^\ell}| = 1_{W_\ell}$ for some measurable set $W_\ell \subset \mathbb{R}^N$ and $\ell = 1, \ldots, L$. Such a wavelet is called a minimally supported frequency, or MSF, wavelet. We would like to spend a few moments to explain the appropriateness of this moniker. In doing so, we follow the example of [18], Chapter 2.

Lemma II.1.1. Given $f \in L^2(\mathbb{R})$, the family $\{T_k f : k \in \mathbb{Z}^N\}$ is orthonormal if and only if for almost every $\xi \in \mathbb{R}^N$ we have

$$\sum_{k \in \mathbb{Z}^N} |\hat{f}(\xi + k)|^2 = 1.$$

Proof. We use the fact that

$$\overline{T_k f}(\xi) = \int_{\mathbb{R}^N} T_k f(x) e^{-2\pi i (\xi, x)} dx = \int_{\mathbb{R}^N} f(x - k) e^{-2\pi i (\xi, x)} dx$$

$$= \int_{\mathbb{R}^N} f(x) e^{-2\pi i (\xi, x + k)} dx = \int_{\mathbb{R}^N} f(x) e^{-2\pi i (\xi, x)} e^{-2\pi i (\xi, k)} dx$$

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\[ = e^{-2\pi i \langle \xi, k \rangle} \int_{\mathbb{R}^N} f(x) e^{-2\pi i \langle \xi, x \rangle} dx = \hat{f}(\xi) e^{-2\pi i \langle \xi, k \rangle}, \]

which gives

\[ \int_{\mathbb{R}^N} f(x) \overline{T_k f(x)} dx = \int_{\mathbb{R}^N} \hat{f}(\xi) \overline{T_k \hat{f}(\xi)} d\xi \]

\[ = \int_{\mathbb{R}^N} \hat{f}(\xi) \overline{\hat{f}(\xi)} e^{2\pi i \langle \xi, k \rangle} d\xi = \int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 e^{2\pi i \langle \xi, k \rangle} d\xi \]

\[ = \sum_{m \in \mathbb{Z}^N} \int_{\mathbb{T}^N + m} |\hat{f}(\xi + m)|^2 e^{2\pi i \langle \xi + m, k \rangle} d\xi = \sum_{m \in \mathbb{Z}^N} \int_{\mathbb{T}^N} |\hat{f}(\xi + m)|^2 e^{2\pi i \langle \xi, k \rangle} d\xi. \quad \text{(II.1)} \]

If \( \{T_k f : k \in \mathbb{Z}^N\} \) is orthonormal, then (II.1) yields

\[ \delta_{k,0} = \int_{\mathbb{R}^N} f(x) \overline{T_k f(x)} dx = \int_{\mathbb{T}^N} \sum_{m \in \mathbb{Z}^N} |\hat{f}(\xi + m)|^2 e^{2\pi i \langle \xi, k \rangle} d\xi. \quad \text{(II.2)} \]

Let \( F \) be the \( \mathbb{Z}^N \)-periodic function given by

\[ F(\xi) = \sum_{m \in \mathbb{Z}^N} |\hat{f}(\xi + m)|^2 \]

and let \( c_n(F) \) be the \( n^{\text{th}} \) Fourier coefficient of \( F \). That is
\[ c_n(F) = \int_{\mathbb{T}^N} F(\xi) e^{-2\pi i (\xi, n)} d\xi. \]

Then (II.2) shows that \( c_n(F) = \delta_{-n,0} \). That is, \( c_0(F) = 1 \) and \( c_n(F) = 0 \) for all \( n \neq 0 \). We claim that \( F(\xi) = 1 \) for almost every \( \xi \in \mathbb{R}^N \). Indeed, consider the constant function \( \bar{F} \equiv 1 \). Direct calculation yields

\[ c_n(\bar{F}) = \int_{\mathbb{T}^N} e^{-2\pi i (\xi, n)} d\xi = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases} \]

Thus, by the uniqueness of Fourier coefficients of functions in \( L^1(\mathbb{T}^N) \) it follows that \( F(\xi) = \bar{F}(\xi) \) for almost every \( \xi \in \mathbb{T}^N \).

On the other hand, if \( \sum_{m \in \mathbb{Z}^N} |\hat{f}(\xi + m)|^2 = 1 \) for almost every \( \xi \in \mathbb{R}^N \), then (II.1) yields

\[ \int_{\mathbb{R}^N} f(x) \overline{T_k f(x)} dx = \int_{\mathbb{T}^N} \sum_{m \in \mathbb{Z}^N} |\hat{f}(\xi + m)|^2 e^{2\pi i (\xi, k)} d\xi = \int_{\mathbb{T}^N} e^{2\pi i (\xi, k)} d\xi = \delta_{k,0}, \]

which implies that \( \{T_k f : k \in \mathbb{Z}^N\} \) is orthonormal.

\[ \square \]

**Theorem II.1.2.** If \( \{T_k f : k \in \mathbb{Z}^N\} \subset L^2(\mathbb{R}^N) \) is an orthonormal family of functions, then \( |\text{supp}(\hat{f})| \geq 1 \) with equality if and only if \( \hat{f} = 1_{W_f} \) for some measurable set \( W_f \subset \mathbb{R}^N \) with \( |W_f| = 1 \).

**Proof.** Suppose that \( \{T_k f : k \in \mathbb{Z}^N\} \subset L^2(\mathbb{R}^N) \) is an orthonormal family of functions. First note that \( \sum_{k \in \mathbb{Z}^N} |\hat{f}(\xi + k)|^2 = 1 \) for almost every \( \xi \in \mathbb{R}^N \) by
Lemma II.1.1 and, hence, \( |\hat{f}(\xi)| \leq 1 \) for almost every \( \xi \in \mathbb{R}^N \). Thus, we have

\[
|\text{supp}(\hat{f})| = \int_{\text{supp}(\hat{f})} 1 \, d\xi \geq \int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 \, d\xi = \|\hat{f}\|^2 = \|f\|^2 = 1.
\]

Therefore, all that remains is to show that \( |\text{supp}(\hat{f})| = 1 \) if and only if \( |\hat{f}| = 1_{W_f} \) for some measurable set \( W_f \subset \mathbb{R}^N \) with \( |W_f| = 1 \). Indeed, if \( |\hat{f}| = 1_{W_f} \) then \( |W_f| = |\text{supp}(\hat{f})| = \|\hat{f}\|^2 = \|f\|^2 = 1 \). On the other hand, suppose that \( |\text{supp}(\hat{f})| = 1 \) and let \( X = \{\xi \in \mathbb{R}^N : |\hat{f}(\xi)| < 1\} \). If \( |X| > 0 \), then we have

\[
1 = \|f\|^2 = \|\hat{f}\|^2 = \int_{\text{supp}(\hat{f})} |\hat{f}(\xi)|^2 \, d\xi < |\text{supp}(\hat{f}) \setminus X| + |X| = |\text{supp}(\hat{f})| = 1.
\]

Since this is a contradiction, it follows that \( |X| = 0 \). Thus, \( |\hat{f}(\xi)| = 1 \) for almost every \( \xi \in \text{supp}(\hat{f}) \) and, therefore, \( |\hat{f}| = 1_{\text{supp}(\hat{f})} \).

Therefore, if \( \Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^N) \) is a wavelet, then \( \sum_{\ell=1}^L |\text{supp}(\hat{\psi}^\ell)| \geq L \). Furthermore, \( \sum_{\ell=1}^L |\text{supp}(\hat{\psi}^\ell)| \) reaches this minimum value of \( L \) if and only if for each \( \ell = 1, \ldots, L \) we have \( |\hat{\psi}^\ell| = 1_{W_\ell} \) for some measurable set \( W_\ell \subset \mathbb{R}^N \). Since the domain of \( \hat{f} \) is commonly referred to as the frequency domain of \( f \), the choice of names for this type of wavelets is a natural one.
II.2 Wavelet Sets

If $\Psi = \{\psi^1, \ldots, \psi^L\}$ is an MSF wavelet with $|\hat{\psi^\ell}| = 1_{W^\ell}$ for each $\ell = 1, \ldots, L$, we denote the union of the supports of the Fourier transforms by

$$W = \bigcup_{\ell=1}^{L} W^\ell$$

and we refer to $W$ as a wavelet set (of order $L$). Clearly, the existence of an MSF wavelet implies the existence of a wavelet set. However, it is a fact that wavelet sets can be identified directly by intrinsic properties independent of an associated MSF wavelet. This can be done by the use of the following characterization of wavelet sets (see [11], Theorem 2.6):

**Theorem II.2.1.** A measurable set $W \subset \mathbb{R}^N$ is a wavelet set of order $L$ if and only if the following two conditions are satisfied:

$$\sum_{k \in \mathbb{Z}^N} 1_W(\xi + k) = L \text{ for a.e. } \xi \in \mathbb{R}^N$$  \hspace{1cm} (II.3)

$$\sum_{j \in \mathbb{Z}} 1_W((A^*)^j \xi) = 1 \text{ for a.e. } \xi \in \mathbb{R}^N.$$  \hspace{1cm} (II.4)

Although Theorem II.2.1 allows us to characterize wavelet sets without relying on an associated MSF wavelet, its proof does rely heavily on the following characterization of MSF wavelets. *(Note that in addition to the following theorem in [11], Theorem 2.4, a similar characterization appears in [15].)*
Theorem II.2.2. Let $W_1, \ldots, W_L$ be a collection of measurable sets in $\mathbb{R}^N$, and for each $\ell = 1, \ldots, L$ let $\psi^{\ell} \in L^2(\mathbb{R}^N)$ be such that $|\hat{\psi}^{\ell}| = 1_{W_\ell}$. Then $\Psi = \{\psi^1, \ldots, \psi^L\}$ is a wavelet associated with the dilation $A \in M_N(\mathbb{R})$ if and only if the following two conditions are satisfied:

$$\sum_{k \in \mathbb{Z}^N} 1_{W_\ell}(\xi + k)1_{W_{\ell'}}(\xi + k) = \delta_{\ell, \ell'} \quad \text{for a.e. } \xi \in \mathbb{R}^N \text{ and all } \ell, \ell' = 1, \ldots, L \quad \text{(II.5)}$$

$$\sum_{j \in \mathbb{Z}} \sum_{\ell = 1}^L 1_{W_\ell}(A^j \xi) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^N. \quad \text{(II.6)}$$

Proof. We have already shown in Lemma II.1.1 that for each $\ell = 1, \ldots, L$ the family of functions $\{T_k \psi^{\ell} : k \in \mathbb{Z}^N\}$ is orthonormal if and only if $\sum_{k \in \mathbb{Z}^N} |\hat{\psi}^{\ell}(\xi + k)|^2 = 1$ for a.e. $\xi \in \mathbb{R}^N$, which is the same as

$$\sum_{k \in \mathbb{Z}^N} 1_{W_\ell}(\xi + k) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^N. \quad \text{(II.7)}$$

Therefore, the family $\{T_k \psi^{\ell} : k \in \mathbb{Z}^N; \ell = 1, \ldots, L\}$ is orthonormal if and only if (II.7) holds for each $\ell = 1, \ldots, L$ and the sets $W_\ell$ are pairwise disjoint (modulo null sets), which is equivalent to (II.5). Furthermore, the family $\{D_j T_k \psi^{\ell} : j \in \mathbb{Z}; k \in \mathbb{Z}^N; \ell = 1, \ldots, L\}$ is orthonormal if and only if $\{T_k \psi^{\ell} : j \in \mathbb{Z}; \ell = 1, \ldots, L\}$ is orthonormal and the families $\{D_j T_k \psi^{\ell} : k \in \mathbb{Z}^N; \ell = 1, \ldots, L\}$ are mutually orthogonal with respect to $j \in \mathbb{Z}$. 

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Now, since \( \text{span} \{ T_k \psi^\ell : k \in \mathbb{Z}^N \} = \{ f \in L^2(\mathbb{R}^N) : \text{supp}(\hat{f}) \subset W_\ell \} \) for each \( \ell = 1, \ldots, L \), then

\[
\text{span} \{ T_k \psi^\ell : k \in \mathbb{Z}^N; \ell = 1, \ldots, L \} = \left\{ f \in L^2(\mathbb{R}^N) : \text{supp}(\hat{f}) \subset \bigcup_{\ell=1}^L W_\ell \right\}.
\]

We denote this set by \( S_0 \). Then for any \( j \in \mathbb{Z} \) we have

\[
\text{span} \{ D_j T_k \psi^\ell : k \in \mathbb{Z}^N; \ell = 1, \ldots, L \} = \left\{ f \in L^2(\mathbb{R}^N) : \text{supp}(\hat{f}) \subset (A^j)^{\ell} \left( \bigcup_{\ell=1}^L W_\ell \right) \right\}.
\]  
(II.8)

We denote each of these sets by \( S_j \). Clearly the affine system associated with \((\Psi, A)\) is a basis for \( L^2(\mathbb{R}^N) \) if and only if

\[
\text{span} \bigcup_{j \in \mathbb{Z}} \{ D_j T_k \psi^\ell : k \in \mathbb{Z}^N; \ell = 1, \ldots, L \} = L^2(\mathbb{R}^N),
\]

Combining this with the preceding argument about orthonormality, we see that the affine system associated with \((\Psi, A)\) is an orthonormal basis for \( L^2(\mathbb{R}^N) \) if and only if \( \{ T_k \psi^\ell : j \in \mathbb{Z}; \ell = 1, \ldots, L \} \) is orthonormal and

\[
\bigoplus_{j \in \mathbb{Z}} \text{span} \{ D_j T_k \psi^\ell : k \in \mathbb{Z}^N; \ell = 1, \ldots, L \} = L^2(\mathbb{R}^N),
\]  
(II.9)

which is to say that \( \bigoplus_{j \in \mathbb{Z}} S_j = L^2(\mathbb{R}^N) \). By (II.8), we see that (II.9) is equivalent to

\( \{ (A^j)^{\ell} \left( \bigcup_{\ell=1}^L W_\ell \right) : j \in \mathbb{Z} \} \) partitioning \( \mathbb{R}^N \) (modulo null sets), which is equivalent to (II.6).
Therefore, the affine system associated with $(\Psi, A)$ is a wavelet if and only if both (II.5) and (II.6) hold.

\[\square\]

In the proof of Theorem II.2.1, we will need to make use of the \textit{translation projection} operator $\tau$ on $\mathbb{R}^N$ given by the fact that for each $\xi \in \mathbb{R}^N$ we have $\tau(\xi) \in \mathbb{T}^N$ with $\xi - \tau(\xi) \in \mathbb{Z}^N$. We will also need to make use of the following claim.

\textbf{Claim II.2.3.} For every measurable set $\tilde{X} \subset \mathbb{R}^N$ there is a measurable set $X \subset \tilde{X}$ such that $\tau|_X$ is injective and $\tau(X) = \tau(\tilde{X})$.

\textit{Proof.} We construct $X$ inductively. Let $z_1, z_2, \ldots$ be an enumeration of $\mathbb{Z}^N$. Let

$$S_1 = \tilde{X} \cap (\mathbb{T}^N + z_1)$$

and for any $n > 1$ let

$$S_n = (\tilde{X} \cap (\mathbb{T}^N + z_n)) \setminus \bigcup_{k=1}^{n-1} (S_k - z_k + z_n).$$

Now define $X = \bigcup_{n=1}^{\infty} S_n$.

Notice that $X$ is measurable and contained in $\tilde{X}$ by construction.

To see that $\tau|_X$ is injective, suppose that $x, y \in X$ such that $\tau(x) = \tau(y)$. Then there exist $m, n \in \mathbb{N}$ such that $x \in S_m$ and $y \in S_n$ and, hence, $x - z_m = \tau(x) = \tau(y) = y - z_n$. Notice that $y \in X$ implies $m \geq n$ since
\(y = x - z_m + z_n\). Similarly, \(x \in X\) implies \(m \leq n\) since \(x = y - z_n + z_m\). Thus, \(m = n\). Therefore, \(x = y\).

Finally, to see that \(\tau(X) = \tau(\tilde{X})\) we first note that \(\tau(X) \subset \tau(\tilde{X})\) since \(X \subset \tilde{X}\). Thus, it remains only to show that \(\tau(X) \supset \tau(\tilde{X})\). Suppose that \(x \in \tau(\tilde{X})\).

Then \(\{n \in \mathbb{N} : x + z_n \in \tilde{X}\} \neq \emptyset\). Let \(k = \min\{n \in \mathbb{N} : x + z_n \in \tilde{X}\}\) and let \(y = x + z_k\). Then \(y \in \tilde{X} \cap (\mathbb{T}^N + z_k)\) and for all \(n < k\) we have \(y - z_k + z_n \notin \tilde{X}\). Thus, \(y \in S_n \subset X\). Furthermore, \(x = \tau(y)\). Therefore, \(x \in \tau(X)\).

\(\Box\)

We are now ready to prove Theorem II.2.1.

**Proof.** Suppose \(W\) is a wavelet set of order \(L\). Then there exists an MSF wavelet \(\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^N)\) such that \(\hat{\psi}^\ell = \mathbf{1}_{\text{supp}(\psi^\ell)}\) for each \(\ell = 1, \ldots, L\) and \(W = \bigcup_{\ell=1}^L W_\ell\) where \(W_\ell = \text{supp}(\psi^\ell)\). Since \(W_\ell = \text{supp}(\psi^\ell)\), we have \(\mathbf{1}_W = \sum_{\ell=1}^L \mathbf{1}_{W_\ell}\). By (II.5) we have that \(\sum_{k \in \mathbb{Z}^N} \mathbf{1}_{W_\ell}(\xi + k) = 1\) for almost every \(\xi \in \mathbb{R}^N\) and each \(\ell = 1, \ldots, L\). Thus

\[
\sum_{k \in \mathbb{Z}^N} \mathbf{1}_W(\xi) = \sum_{k \in \mathbb{Z}^N} \sum_{\ell=1}^L \mathbf{1}_{W_\ell}(\xi + k) = \sum_{\ell=1}^L \sum_{k \in \mathbb{Z}^N} \mathbf{1}_{W_\ell}(\xi + k) = L \quad \text{for a.e.} \quad \xi \in \mathbb{R}^N,
\]

giving (II.3). Furthermore, by (II.6) we have \(\sum_{j \in \mathbb{Z}} \sum_{\ell=1}^L \mathbf{1}_{W_\ell}((A^t)^j \xi) = 1\) for almost every \(\xi \in \mathbb{R}^N\). Thus

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\[
\sum_{j \in \mathbb{Z}} \mathbbm{1}_W((A^T)^j \xi) = \sum_{j \in \mathbb{Z}} \sum_{\ell=1}^{L} \mathbbm{1}_{W_{\ell}}((A^T)^j \xi) = 1 \text{ for a.e. } \xi \in \mathbb{R}^N,
\]
giving (II.4).

Now suppose, on the other hand, that \(W\) satisfies (II.3) and (II.4). Condition (II.3) implies that for almost every \(\xi \in \mathbb{T}^N\), there are \(L\) points \(\xi_1, \ldots, \xi_L \in W\) such that \(\xi_\ell - \xi \in \mathbb{Z}^N\) (which is to say that \(\tau(\xi_\ell) = \xi\)) for each \(\ell = 1, \ldots, L\). In particular, this means that \(\tau(W) = \mathbb{T}^N\) (modulo null sets). Thus by the above claim regarding \(\tau\) there exists a measurable set \(W_1 \subset W\) such that \(\tau|_{W_1}\) is injective and \(\tau(W_1) = \mathbb{T}^N\) (modulo null sets). Suppose that for some \(n = 1, \ldots, L - 1\) we have a collection of pairwise disjoint measurable sets \(W_1, \ldots, W_n \subset W\) such that for each \(\ell = 1, \ldots, n\) we have \(\tau|_{W_\ell}\) is injective and \(\tau(W_\ell) = \mathbb{T}^N\) (modulo null sets). Then

\[
\sum_{k \in \mathbb{Z}^N} \mathbbm{1}_{W \cup \bigcup_{\ell=1}^{n} W_{\ell}}(\xi + k) = L - n \text{ a.e. } \xi \in \mathbb{R}^N.
\]

Thus, by our claim about \(\tau\) once again, there exists a measurable set \(W_{n+1} \subset W \setminus \bigcup_{\ell=1}^{n} W_{\ell}\) such that \(\tau|_{W_{n+1}}\) is injective and \(\tau(W_{n+1}) = \mathbb{T}^N\) (modulo null sets). Hence, we have a collection of pairwise disjoint measurable sets \(\{W_1, \ldots, W_i\}\) that partition \(W\) and each have the property that
Thus, (II.5) holds. Furthermore, since $W = \bigcup_{\ell=1}^{L} W_{\ell}$, then

\[
\sum_{j \in \mathbb{Z}} \sum_{\ell=1}^{L} \mathbb{I}_{W_{\ell}}((A^T)^j \xi) = \sum_{j \in \mathbb{Z}} \mathbb{I}_{W}((A^T)^j \xi) = 1 \text{ for a.e. } \xi \in \mathbb{R}^N
\]

by (II.4), giving (II.6).

In the next section we provide a formula for all wavelet sets in $\mathbb{R}$ that consist of three intervals.

II.3 A Characterization of Two & Three Interval Wavelet Sets in $\mathbb{R}$

II.3.1 Construction and Characterization Of Two Interval Wavelet Sets

To construct a wavelet set in $\mathbb{R}$, we employ Theorem II.2.1 which in this case states that a measurable set $W \subset \mathbb{R}$ is a wavelet set associated with the dilation $a > 1$ if and only if
\[ \sum_{k \in \mathbb{Z}} 1_{W}(\xi + k) = 1 \text{ for a.e. } \xi \in \mathbb{R} \quad (\text{II.10}) \]

\[ \sum_{j \in \mathbb{Z}} 1_{W}(a^j \xi) = 1 \text{ for a.e. } \xi \in \mathbb{R} \quad (\text{II.11}) \]

Notice that (II.11) implies that \( W \) cannot contain any intervals (of positive length) containing 0. Indeed, if \( I \subset W \) is an interval with \( 0 \in I \), then for every \( \xi \in \mathbb{R} \) we would have \( \xi \in a^j I \) for infinitely many \( j \in \mathbb{Z} \) and, hence,

\[ \sum_{j \in \mathbb{Z}} 1_{W}(a^j \xi) = \infty \text{ for all } \xi \in \mathbb{R}. \]

Furthermore, it also implies that \( W \) must have at least one negative component of positive measure and at least one positive component of positive measure. If, for instance, \( |W \cap (-\infty, 0)| = 0 \), then we would have \( \sum_{j \in \mathbb{Z}} 1_{W}(a^j \xi) = 0 \) for all \( \xi < 0 \). Thus, if we wish to construct a wavelet set \( W \) that is composed of exactly two intervals, we must have \( W = [b, c] \cup [d, e] \) with \( b < c < 0 < d < e \). Of course, the intervals do not have to be closed, but we choose to adhere to convention. Clearly, (II.11) is equivalent to \([b, c] \) and \([d, e]\) partitioning \((-\infty, 0)\) and \((0, \infty)\), respectively, by dilations (modulo null sets) as shown in Figure II.1. Note that in Figures II.1 & II.2 we have \( I_1 = [b, c] \) and \( I_2 = [d, e] \).

On the other hand, (II.10) is equivalent to the fact that \( d \) is an integer shift of \( c \) and the lengths of the two intervals sum to 1, as shown in Figure II.2.

In other words, (II.11) is equivalent to \( b = ac \) and \( e = ad \), while (II.10) is equivalent to \( (a - 1)(d - c) = 1 \) and \( d = c + n \) for some \( n \in \mathbb{N} \) with \( c + n > 0 \). Solving for \( a \) yields \( a = \frac{n+1}{n} \).
Figure II.1: Dilation partition condition for two intervals.

Figure II.2: Translation partition condition for two intervals.

These results can be summarized in the following theorem:

**Theorem II.3.1.** There exist two interval wavelet sets corresponding to the dilation $a$ if and only if $a = \frac{n+1}{n}$ for some $n \in \mathbb{N}$. Furthermore, if $a = \frac{n+1}{a}$ for some $n \in \mathbb{N}$, then $W$ is a two interval wavelet set corresponding to $a$ if and only if:

$$W = [ax, x] \cup [x + \frac{1}{a-1}, ax + \frac{a}{a-1}]$$

for some $x \in \left(\frac{a-1}{a-1}, 0\right)$. 

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Furthermore, the following to corollaries are immediate consequences of the conditions placed on \( a \) and \( x \) in Theorem II.3.1:

**Corollary II.3.2.** There exist no two interval wavelet sets associated with any dilation \( a > 2 \).

**Corollary II.3.3.** If \( a = \frac{n+1}{n} \) for some \( n \in \mathbb{N} \), then there are uncountably many two interval wavelet sets corresponding to the dilation \( a \).

### II.3.2 Construction Of Three Interval Wavelet Sets

Let us now move to wavelet sets consisting of three intervals. In [7] Bownik gives an example of Speegle’s which provides a formula for a family of wavelet sets in \( \mathbb{R} \) consisting of three intervals and depending on the dilation \( a > 1 \) (see [7], Remark 3). In this section, we extend Speegle’s example to a more general form, characterizing all wavelet sets in \( \mathbb{R} \) consisting of three intervals. Furthermore, we show that there are countably many such wavelet sets associated with each dilation \( a > 1 \).

We consider \( W = [b, c] \cup [d, e] \cup [f, g] \) where \( b < c < 0 < d < e < f < g \). This is sufficient since \( W \) satisfies (II.10) and (II.11) if and only if \(-W\) does. Notice that in the case of three intervals (II.11) implies a slightly more complicated relationship. This condition is satisfied if and only if \([b, c]\) partitions \((-\infty, 0)\) by dilations (modulo null sets) and \([d, e]\) and \([f, g]\) partition \((0, \infty)\) by dilations (modulo null sets) in an interlacing pattern as shown in Figure II.3. *Note that in Figures II.3–II.5*

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we have $I_1 = [b, c]$, $I_2 = [d, e]$, and $I_3 = [f, g]$. Note also that $p$ is some integer greater than 0. We refer to it as the interlacing parameter.

Figure II.3: Dilation partition condition for three intervals.

On the other hand, the relationship implied by (II.10) is similar to the two interval case, with the exception that there are now two ways in which it can be satisfied. These are shown in Figures II.4 and II.5.

Figure II.4: Translation partition condition for three intervals. (Option 1)
Figure II.5: Translation partition condition for three intervals. (Option 2)

Initially we will concern ourselves only with three interval wavelet sets that satisfy the translation partition condition as shown in Figure II.4. Let us construct all such wavelet sets. Notice that each one represents a solution to the following system of equations for some set of values $m, n, p \in \mathbb{N}$.

\[
\begin{align*}
    m &= g - d \\
    n &= e - b \\
    1 &= c - b + e - d + g - f \\
    0 &= -b + ac \\
    0 &= a^p e - f \\
    0 &= a^{p+1} d - g
\end{align*}
\]  

(II.12)

For each fixed set of values $m, n, p \in \mathbb{N}$, (II.12) has a solution given by
$$[b,c] = \left[ \frac{a(m-n(a^p-1)-1)}{a^{p+1}-1}, \frac{m-n(a^p-1)-1}{a^{p+1}-1} \right]$$

$$[d,e] = \left[ \frac{m}{a^{p+1}-1}, \frac{a(m-1)+n(a-1)}{a^{p+1}-1} \right]$$  \hspace{1cm} \text{(II.13)}$$

$$[f,g] = \left[ \frac{a^p(a(m-1)+n(a-1))}{a^{p+1}-1}, \frac{a^{p+1}m}{a^{p+1}-1} \right]$$

Furthermore, the conditions $b < c < 0 < d < e < f < g$ impose the added conditions on $m, n$:

$$0 < n < \frac{a}{a-1}$$ \hspace{1cm} \text{(II.14)}$$

$$\frac{a}{a-1} - n < m < n(a^p-1) + 1$$ \hspace{1cm} \text{(II.15)}$$

Next we will investigate three interval wavelet sets that satisfy the translation partition condition as shown in Figure II.5. Each of these sets represents a solution to the following system of equations for some set of values $m, n, p \in \mathbb{N}$. 

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\[
\begin{align*}
\begin{cases}
m &= f - e \\
n &= d - c \\
1 &= c - b + e - d + g - f \\
0 &= -b + ac \\
0 &= a^p e - f \\
0 &= a^{p+1} d - g
\end{cases}
\end{align*}
\] (II.16)

For each fixed set of values \( m, n, p \in \mathbb{N} \), (II.16) has a solution given by

\[
[b, c] = \left[ \frac{m - n(a^{p+1} - 1) + 1}{a^p - 1}, \frac{m - n(a^{p+1} - 1) + 1}{a(a^p - 1)} \right]
\]

\[
[d, e] = \left[ \frac{m - n(a - 1) + 1}{a(a^p - 1)}, \frac{m}{a^p - 1} \right]
\] (II.17)

\[
[f, g] = \left[ \frac{a^p m}{a^p - 1}, \frac{a^p(m - n(a - 1) + 1)}{a^p - 1} \right]
\]

and the corresponding conditions on \( m, n \) are

\[
0 < n < \frac{1}{a - 1}
\] (II.18)

\[
\frac{1}{a - 1} - n < m < n(a^{p+1} - 1) - 1
\] (II.19)
II.3.3 Analysis of the Feasible Regions

In (II.14) and (II.15) we are given a feasible region for \((m, n)\) which depends only on the parameters \(a\) and \(p\). We will denote this by \(\mathcal{F}_1(a, p)\). One notices that \(\mathcal{F}_1(a, p)\) is the interior of the triangle shown in Figure II.6.

\[
\begin{align*}
\partial_1(m) &= \frac{a}{a-1} \\
\partial_2(m) &= \frac{a}{a-1} - m \\
\partial_3(m) &= \frac{m-1}{a^{p-1}} \\
\end{align*}
\]

Figure II.6: \(\mathcal{F}_1(a, p)\). Feasible region defined by (II.14) & (II.15).

Since (II.13) only gives us a wavelet set for integers \(n, m\) satisfying (II.14) and (II.15), it is of interest to know for what values of \(a, p\) is the intersection of \(\mathcal{F}_1(a, b)\) with \(\mathbb{Z}^2\) nonempty.

First we consider the case when \(\frac{a}{a-1} \notin \mathbb{Z}\). That is, \(a \neq \frac{k+1}{k}\) for all \(k \in \mathbb{N}\). In this case, it is easy to see that \(\mathcal{F}_1(a, p) \cap \mathbb{Z}^2 \neq \emptyset\) for all \(p \in \mathbb{N}\). Indeed, we clearly have \(\lfloor \frac{a}{a-1} \rfloor < \frac{a}{a-1} \) and \(\partial_2(1) = \frac{a}{a-1} - 1 < \lfloor \frac{a}{a-1} \rfloor \). Furthermore, \(\partial_3(1) = 0 < \lfloor \frac{a}{a-1} \rfloor \).

Therefore, \((1, \lfloor \frac{a}{a-1} \rfloor) \in \mathcal{F}_1(a, p)\) regardless of \(p\). Note, however, that when \(p = 1\) we have \(\partial_1(2) - \partial_3(2) = 1\) in addition to \(\partial_1(1) - \partial_2(1) = 1\). Thus, \((1, \lfloor \frac{a}{a-1} \rfloor)\) and \((2, \lceil \frac{a}{a-1} \rceil)\) are the only points in \(\mathcal{F}_1(a, 1)\).
Suppose now that \( a = \frac{k+1}{k} \) for some \( k \in \mathbb{N} \) and, hence, \( \frac{a}{a-1} \in \mathbb{Z} \). Notice that if \( p = 1 \) then \( \mathcal{F}_1(a, p) \) becomes as shown in Figure II.7.

![Diagram](image)

Figure II.7: \( \mathcal{F}_1(a, 1) \).

In this case, we have \( \frac{a}{a-1} - 1 = \partial_2(1) \) and \( \frac{a}{a-1} - 1 = \partial_3(2) \) since \( \frac{a}{a-1} - 1 = \frac{1}{a-1} \). Hence, \((1, n) \notin \mathcal{F}_1(a, 1)\) and \( (2, n) \notin \mathcal{F}_1(a, 1) \) for any \( n \in \mathbb{Z} \).

Furthermore, for all \( m \in \mathbb{Z} \) with \( 3 \leq m \leq a + 1 \) we have \( \partial_1(m) - \partial_3(m) \leq \partial_1(2) - \partial_3(2) = 1 \) and, hence, \((m, n) \notin \mathcal{F}_1(a, 1) \) for any \( 3 \leq m \leq a + 1 \) and \( n \in \mathbb{Z} \). Therefore, \( \mathcal{F}(a, 1) \cap \mathbb{Z}^2 = \emptyset \). However, if \( p > 1 \) then we have \( \partial_3(2) = \frac{1}{a^p - 1} < \frac{1}{a-1} = \frac{a}{a-1} - 1 \) and so in this case we have \( (2, \lceil \frac{1}{a-1} \rceil) \in \mathcal{F}_1(a, p) \).

We can summarize this discussion in the following statement:

\[
\mathcal{F}_1(a, p) \cap \mathbb{Z}^2 \begin{cases} 
= \emptyset \text{ if } a = \frac{k+1}{k} \text{ for some } k \in \mathbb{N} \text{ and } p = 1 \\
\neq \emptyset \text{ otherwise}
\end{cases}
\]

Another interesting property regarding \( \mathcal{F}_1(a, p) \cap \mathbb{Z}^2 \) is the critical role that the value \( a = 2 \) plays. Notice that as \( a \) takes on larger values, \( \frac{a}{a-1} \) approaches 1.
Thus, the boundary $\partial_1$ quickly becomes restrictive from above. In fact, for all values of $a > 2$ we have $\frac{a}{a-1} < 2$ and so the only values of $(m, n) \in \mathcal{F}_1(a, p) \cap \mathbb{Z}^2$ that remain are $(m, 1)$ where $1 \leq m < a^p$, as shown in Figure II.8.

![Figure II.8: $\mathcal{F}_1(a, p)$ for $a > 2$.](image)

Similarly, if $a = 2$ the only values of $(m, n) \in \mathcal{F}_1(a, p) \cap \mathbb{Z}^2$ that remain are $(m, 1)$ where $2 \leq m \leq 2^p - 1$, as shown in Figure II.9.

On the other hand, conditions (II.18) and (II.19) give us a feasible region for $(m, n)$ pictured as the interior of the triangle in Figure II.10. We wish to know for what values of $a$ and $p$ do we have $\mathcal{F}_2(a, p) \cap \mathbb{Z}^2 \neq \emptyset$.

Notice that the value $a = 2$ plays an even more critical role this time. Indeed, if $a \geq 2$, then $\frac{1}{a-1} \leq 1$ and so $\mathcal{F}_2(a, p) \cap \mathbb{Z}^2 = \emptyset$. Therefore, we can only have wavelet sets of this type for dilations $1 < a < 2$.

Let us now investigate the case $p = 1$ separately. Notice that when $p = 1$ we have $\mathcal{F}_2(a, 1)$ as in Figure II.11.

Because $\frac{1}{a} < 1$ and the slope of $\partial_3$ is positive, it is clear that $\mathcal{F}_2(a, 1) \cap \mathbb{Z}^2 \neq \emptyset$ if and only if $(1, n) \in \mathcal{F}_2(a, 1)$ for some $n \in \mathbb{N}$. Since
\( \partial_3(m) = \frac{m+1}{a^{p+1} - 1} \)

\( \partial_2(m) = \frac{1}{a - 1} - m \)

\( \partial_1(m) = \frac{1}{a - 1} \)

\( (\frac{a(a-1)}{a-1}, \frac{1}{a-1}) \)

\( (\frac{a^{p-1}}{a^{p(a-1)}}, \frac{1}{a^{p(a-1)}}) \)

\( \partial_1(1) - \partial_3(1) = \frac{1}{a-1} - \frac{2}{a^2-1} = \frac{a-1}{a^2-1} < 1 \), this can only occur for \( n = \lfloor \frac{1}{a-1} \rfloor \). Thus, we wish to characterize the values of \( a \) for which we have \( \frac{2}{a^2-1} < \lfloor \frac{1}{a-1} \rfloor < \frac{1}{a-1} \). With this in mind, let \( n = \lfloor \frac{1}{a-1} \rfloor \). Notice that \( \frac{2}{a^2-1} < n < \frac{1}{a-1} \) if and only if \( \sqrt{1 + \frac{2}{n}} < a < \frac{n+1}{n} \). Therefore, \( (1, n) \in \mathcal{F}_2(a, 1) \) for some \( n \in \mathbb{N} \) if and only if...
\[
\sqrt{1 + \frac{2}{n}} < a < \frac{n+1}{n}, \text{ in which case we have } n = \left\lfloor \frac{1}{a-1} \right\rfloor. \text{ We conclude that } \\
\mathcal{F}_2(a, 1) \cap \mathbb{Z}^2 \neq \emptyset \text{ if and only if } a \in \bigcup_{n \in \mathbb{N}} \left( \sqrt{1 + \frac{2}{n}}, \frac{n+1}{n} \right).
\]

The case \( p \geq 2 \) is most efficiently handled by splitting into two subcases:

1. \( 1 < a \leq \frac{3}{2} \) and \( \frac{3}{2} < a < 2 \). We deal with the second of these cases first since it is far the easiest. Notice that \( \frac{3}{2} < a < 2 \) implies that \( \partial_2(1) < 1 < \partial_1(1) \). Furthermore, for such \( a \) we have that \( p \geq 2 > \log_{\frac{3}{2}}(2) \) implies \( \partial_3(1) = \frac{2}{a^p - 1} < \left(\frac{3}{2}\right)^{2p+1} - 1 < 1 \).

Thus, \( (1, 1) \in \mathcal{F}_2(a, p) \). We conclude that \( \mathcal{F}_2(a, p) \cap \mathbb{Z}^2 \neq \emptyset \) for all \( a \) and \( p \) satisfying \( \frac{3}{2} < a < 2 \) and \( p \geq 2 \).

The last remaining case is \( p \geq 2 \) and \( 1 < a < \frac{3}{2} \). First note that when \( p \geq 2 \) we have \( 1 < \frac{a^p - 1}{a^p(a-1)} \). Thus, \( (1, n) \in \mathcal{F}_2(a, p) \) if and only if \( \partial_2(1) < n < \partial_1(1) \). Since \( \partial_2(1) = \partial_1(1) - 1 \), then this occurs if and only if \( \frac{1}{a-1} \notin \mathbb{Z} \) (which is to say \( a \neq \frac{k+2}{k+1} \) for all \( k \in \mathbb{N} \)). Suppose, then, that \( a = \frac{k+2}{k+1} \) for some \( k \in \mathbb{N} \). Then one wishes to know for what values of \( p \) we have \( (2, \left\lfloor \frac{1}{a-1} \right\rfloor) \in \mathcal{F}_2(a, p) \). Indeed, \( \partial_2(2) = \frac{1}{a-1} - 2 < \left\lfloor \frac{1}{a-1} \right\rfloor < \partial_1(2) \), thus \( (2, \left\lfloor \frac{1}{a-1} \right\rfloor) \in \mathcal{F}_2(a, p) \) whenever,
\( \partial_3(2) < \frac{1}{a-1} - 1 \). We see that this holds whenever \( p > \log_a \left( \frac{2a-1}{2-a} \right) \). But for all \( a \) satisfying \( 1 < a \leq \frac{3}{2} \) we have \( \log_a \left( \frac{2a-1}{2-a} \right) \leq \log_2(4) < 3 \). Therefore, we conclude that \( \mathcal{F}_2(a, p) \cap \mathbb{Z}^2 \neq \emptyset \) for all \( p \geq 3 \) when \( a = \frac{k+2}{k+1} \) for some \( k \in \mathbb{N} \).

We summarize these results below. Compare this with the results for \( \mathcal{F}_1(a, p) \).

\[
\mathcal{F}_2(a, p) \cap \mathbb{Z}^2 \neq \emptyset \quad \text{when} \quad p = 1 \quad \text{if and only if} \quad a \in \bigcup_{n \in \mathbb{N}} \left( \sqrt{1 + \frac{2}{n}} , \frac{n+1}{n} \right)
\]

\[
\text{when} \quad p = 2 \quad \text{if and only if} \quad 1 < a < 2 \quad \text{with} \quad \frac{1}{a-1} \not\in \mathbb{N}
\]

\[
\text{when} \quad p \geq 3 \quad \text{for all} \ a \ \text{satisfying} \quad 1 < a < 2
\]

II.3.4 Characterization Of Three Interval Wavelet Sets

We are now ready to present our main results regarding three interval wavelet sets.

**Theorem II.3.4.** If \( W \subset \mathbb{R} \) is comprised of three intervals, then \( W \) is a wavelet set if and only if \( W \) or \(-W\) is as in (i) or (ii) below:

(i) \[
\left[ \frac{m-n(a^p-1)-1}{a^p-1} , \frac{m-n(a^p-1)-1}{a^p-1} \right] \cup \left[ \frac{m}{a^p+1-1} , \frac{a(m-1)+n(a-1)}{a^p+1-1} \right] \cup \left[ \frac{a^p(a(m-1)+n(a-1))}{a^p+1-1} , \frac{a^p+1m}{a^p+1} \right]
\]

for some \( a > 1, p \in \mathbb{N}, \) and \((n, m) \in \mathbb{Z}^2 \) satisfying

\[
0 < n < \frac{a}{a-1}
\]

\[
\frac{a}{a-1} - n < m < n(a^p - 1) + 1.
\]
\[ i.i \left[ \frac{m-n(a^{p+1} - 1) + 1}{a^{p+1} - 1}, \frac{m-n(a^{p+1} - 1) + 1}{a^{p+1} - 1} \right] \cup \left[ \frac{m-n(a-1) + 1}{a^{p-1} - 1}, \frac{m}{a^{p-1} - 1} \right] \cup \left[ \frac{a^{p}m}{a^{p-1} - 1}, \frac{a^{p}(m-n(a-1) + 1)}{a^{p-1} - 1} \right] \]

for some \( 1 < a < 2, p \in \mathbb{N}, \) and \((n,m) \in \mathbb{Z}^2\) satisfying

\[
0 < n < \frac{1}{a-1} < \frac{1}{a-1} - n < m < n(a^{p+1} - 1) + 1.
\]

**Proof.** This is a direct consequence of the fact that \(W\) is a wavelet set if and only if \(-W\) is a wavelet set, and the fact that the matrices

\[
\begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 1 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
-1 & 1 & -1 & 1 & -1 & 1 \\
-1 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a^p & -1 & 0 \\
0 & 0 & a^{p+1} & 0 & 0 & -1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 0 & 0 & -1 & 1 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
-1 & 1 & -1 & 1 & -1 & 1 \\
-1 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a^p & -1 & 0 \\
0 & 0 & a^{p+1} & 0 & 0 & -1
\end{bmatrix}
\]

are nonsingular for all \( a > 1\), making the solutions to (II.12) and (II.16) unique.

\[ \square \]

**Corollary II.3.5.** If \( a > 2 \), the only three interval wavelet sets associated with \( a \) are \( \pm W \), where \( W \) is as below:

\[
\begin{bmatrix}
a(m - a^p) & m - a^p \\
\frac{a^{p+1} - 1}{a^{p+1} - 1}, & \frac{a^{p+1} - 1}{a^{p+1} - 1}
\end{bmatrix}
\cup
\begin{bmatrix}
m & am - 1 \\
\frac{a^{p+1} - 1}{a^{p+1} - 1}, & \frac{a^{p+1} - 1}{a^{p+1} - 1}
\end{bmatrix}
\cup
\begin{bmatrix}
a^p(am - 1) & a^{p+1}m \\
\frac{a^{p+1} - 1}{a^{p+1} - 1}, & \frac{a^{p+1} - 1}{a^{p+1} - 1}
\end{bmatrix}
\]

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for some \( p \in \mathbb{N} \) and \( m \in \mathbb{Z} \) such that \( 1 \leq m < a^p \).

Furthermore, the only three interval wavelet sets associated with \( a = 2 \) are \( \pm W \), where \( W \) is as below:

\[
\left[ \frac{2(m - 2^p)}{2^{p+1} - 1}, \frac{m - 2^p}{2^{p+1} - 1} \right] \cup \left[ \frac{m}{2^{p+1} - 1}, \frac{2m - 1}{2^{p+1} - 1} \right] \cup \left[ \frac{2^p(2m - 1)}{2^{p+1} - 1}, \frac{2^{p+1}m}{2^{p+1} - 1} \right]
\]

for some \( p, m \in \mathbb{Z} \) such that \( p \geq 2 \) and \( 2 \leq m \leq 2^p - 1 \).

**Corollary II.3.6.** If \( a = \frac{k+1}{k} \) for some \( k \in \mathbb{N} \), there are no three interval wavelet sets with interlacing parameter \( p = 1 \) associated with \( a \). Furthermore, if the dilation \( a \) is such that \( a \neq \frac{k+1}{k} \) for all \( k \in \mathbb{N} \), then there are at most 6 three interval wavelet sets with interlacing parameter \( p = 1 \) associated with \( a \).

**Proof.** The first statement follows from the fact that both \( T_1(a, 1) \cap \mathbb{Z}^2 = \emptyset \) and \( T_2(a, 1) \cap \mathbb{Z}^2 = \emptyset \) when \( a = \frac{k+1}{k} \) for some \( k \in \mathbb{N} \). To justify the second statement, note that in the preceding section it was shown that when \( a \neq \frac{k+1}{k} \) for all \( k \in \mathbb{N} \), then the only points in \( T_1(a, 1) \cap \mathbb{Z}^2 \) are \((1, \lfloor \frac{a}{a-1} \rfloor)\) and \((2, \lfloor \frac{a}{a-1} \rfloor)\). Thus, the only three interval wavelet sets with interlacing parameter \( p = 1 \) associated with \( a \) the satisfy that translation partition condition as shown in Figure II.4 are \( \pm W \) where \( W \) is either of the following with \( n = \lfloor \frac{a}{a-1} \rfloor \):

\[
\left[ \frac{-na(a^p-1)}{a^{p+1}-1}, \frac{-n(a^p-1)}{a^{p+1}-1} \right] \cup \left[ \frac{1}{a^{p+1}-1}, \frac{n(a-1)}{a^{p+1}-1} \right] \cup \left[ \frac{na(a-1)}{a^{p+1}-1}, \frac{-a^{p+1}}{a^{p+1}-1} \right]
\]
\[
\left[ \frac{a(-n(a^p-1)+1)}{a^p+1-1}, \frac{a^p(a-1)+1}{a^p+1-1} \right] \cup \left[ \frac{2}{a^p+1-1}, \frac{a+a(n(a-1))}{a^p+1-1} \right] \cup \left[ \frac{a^p(a-1)}{a^p+1-1}, \frac{2a^p+1}{a^p+1-1} \right].
\]

If, in addition, \( a \in \left( \sqrt{1 + \frac{2}{k}}, \frac{k+1}{k} \right) \) for some \( k \in \mathbb{N} \), then from Figure II.5 we also have \( \pm W \) where \( W \) is

\[
\left[ \frac{-n(a^p+1)+2}{a^p-1}, \frac{-n(a^p+1)-1}{a^p-1} \right] \cup \left[ \frac{-(a-1)+2}{a^p-1}, \frac{1}{a^p-1} \right] \cup \left[ \frac{a^p}{a^p-1}, \frac{a^p(-n(a-1)+2)}{a^p-1} \right]
\]

with \( n = \left\lfloor \frac{1}{a-1} \right\rfloor \).

\[\Box\]

We end this section with an interesting result that is in contrast to Corollary II.3.3

**Theorem II.3.7.** Given any \( a > 1 \) there are an infinite, but countable, number of three interval wavelet sets associated with \( a \).

**Proof.** Let \( a > 1 \). Since \( \mathcal{F}_1(a,p) \cap \mathbb{Z}^2 \neq \emptyset \) for all but finitely many \( p \in \mathbb{N} \), there are at least countably many three interval wavelet sets associated with \( a \). Furthermore, by (II.14),(II.15) and (II.18),(II.19) there are finitely many wavelet sets associated with \( a \) for each \( p \in \mathbb{N} \). Thus, there are at most countably many such sets.

\[\Box\]
II.4 Generalized Scaling Sets

There exists an alternative to wavelet sets that can be useful in constructing MSF wavelets. These are called generalized scaling sets, which we now define. A set $S \subset \mathbb{R}^N$ is a generalized scaling set (of order $L$) associated with the dilation $A \in M_N(\mathbb{R})$ provided $|S| = \frac{L}{|\det(A)|^{\frac{1}{L}}}$ and $A^T S \setminus S$ is a wavelet set (of order $L$) associated with $A$.

It is evident that from the definition that each generalized scaling set produces a wavelet set and, hence, an MSF wavelet. What is not as obvious, but nonetheless true, is that each wavelet set readily yields a generalized scaling set, as the following classification shows (see [11], Proposition 3.2).

**Proposition II.4.1.** A set $S \subset \mathbb{R}^N$ is a generalized scaling set (of order $L$) associated with the dilation $A \in M_N(\mathbb{R})$ if and only if there is some wavelet set $W$ (of order $L$) associated with $A$ such that

$$S = \bigcup_{j=1}^{\infty} (A^T)^{-j} W. \tag{II.20}$$

**Proof.** Suppose that $S \subset \mathbb{R}^N$ is a generalized scaling set (of order $L$) associated with the dilation $A \in M_N(\mathbb{R})$. Then $A^T S \setminus S$ is a wavelet set (of order $L$) associated with $A$. All that remains to be shown is that $S = \bigcup_{j=1}^{\infty} (A^T)^{-j}(A^T S \setminus S)$. Indeed, by (II.3) we see that
\[
\sum_{k \in \mathbb{Z}^N} 1_{\mathbb{A}^S \setminus S}(\xi + k) = L \quad \text{for a.e. } \xi \in \mathbb{R}^N,
\]

which implies that \(|\mathbb{A}^S \setminus S| = L > 0\) and, hence, \(S \subset \mathbb{A}^S\). Thus, we have

\[
\bigcup_{j=1}^{\infty} (\mathbb{A}^T)^{-j}(\mathbb{A}^S \setminus S) \subset S.
\]

However, notice that the sets \((\mathbb{A}^T)^{-j}(\mathbb{A}^S \setminus S)\) are pairwise disjoint since for any value of \(j \geq 1\) we have

\[
(\mathbb{A}^T)^{-j}(\mathbb{A}^S \setminus S) \cap (\mathbb{A}^T)^{-j+1}(\mathbb{A}^S \setminus S) = \emptyset.
\]

Thus,

\[
\left| \bigcup_{j=1}^{\infty} (\mathbb{A}^T)^{-j}(\mathbb{A}^S \setminus S) \right| = \sum_{j=1}^{\infty} \left| (\mathbb{A}^T)^{-j}(\mathbb{A}^S \setminus S) \right| = \sum_{j=1}^{\infty} \left| \text{det}(A) \right|^{-j} \left| \mathbb{A}^S \setminus S \right| = \frac{|\mathbb{A}^S \setminus S|}{|\text{det}(A)| - 1} = \frac{L}{|\text{det}(A)| - 1} = |S|.
\]

Since \(\bigcup_{j=1}^{\infty} (\mathbb{A}^T)^{-j}(\mathbb{A}^S \setminus S) \subset S\) and \(|\bigcup_{j=1}^{\infty} (\mathbb{A}^T)^{-j}(\mathbb{A}^S \setminus S)| = |S|\), it follows that \(S = \bigcup_{j=1}^{\infty} (\mathbb{A}^T)^{-j}(\mathbb{A}^S \setminus S)\) (modulo null sets).

Suppose, on the other hand, that \(S = \bigcup_{j=1}^{\infty} (\mathbb{A}^T)^{-j}W\) for some wavelet set \(W\) (of order \(L\)) associated with the dilation \(A \in M_N(\mathbb{R})\). By (II.4) we see that the sets \((\mathbb{A}^T)^{-j}W\) are pairwise disjoint with respect to \(j\). Thus,

\[
\mathbb{A}^S \setminus S = \bigcup_{j=0}^{\infty} (\mathbb{A}^T)^{-j}W \setminus \bigcup_{j=1}^{\infty} (\mathbb{A}^T)^{-j}W = W,
\]

and so \(\mathbb{A}^S \setminus S\) is a wavelet set (of order \(L\)) associated with \(A\). Furthermore,
\[ |S| = \left| \bigcup_{j=1}^{\infty} (A^T)^{-j} (A^T S \setminus S) \right| = \frac{L}{|\text{det}(A)| - 1} \] (as shown above). Therefore, \(S\) is a generalized scaling set (of order \(L\)) associated with \(A\).

\[ \square \]

Occasionally, constructing a wavelet with certain desired properties can be done most easily by constructing a generalized scaling set such that the MSF wavelet associated with the generalized scaling set will automatically inherit the properties in question. We will see a use of this strategy later. Thus, it is desirable to have a characterization of generalized scaling sets that does not depend on finding a wavelet set. Bownik, Rzeszotnik, and Speegle have proven such a characterization for integer dilations (see [11], Theorem 3.3). We now prove a generalization of this characterization to allow for rational dilations.

**Theorem II.4.2.** A measurable set \(S \subset \mathbb{R}^N\) is a generalized scaling set (of order \(L\)) associated with the rational \(A \in M_N(\mathbb{Q})\) if and only if the following conditions are satisfied:

\[ S \subset A^T S \] (II.21)

\[ |S| = \frac{L}{|\text{det}(A)| - 1} \] (II.22)
\[
\lim_{n \to \infty} \mathbb{1}_S((A^T)^{-n}\xi) = 1 \quad (II.23)
\]

\[
\sum_{\omega \in \Omega} \sum_{k \in \mathbb{Z}^N} \mathbb{1}_S(\xi + (A^T)^{-\omega} + k) = |\Omega'| \cdot L + \sum_{\omega' \in \Omega'} \sum_{k \in \mathbb{Z}^N} \mathbb{1}_S(A^T\xi + \omega' + k) \quad (II.24)
\]

where \(\Omega, \Omega'\) are transversals of \(\Gamma / A^T\mathbb{Z}^N\) and \(\Gamma / \mathbb{Z}^N\), respectively, with \(\Gamma = \mathbb{Z}^N + A^T\mathbb{Z}^N\), and (II.23) and (II.24) are for almost every \(\xi \in \mathbb{R}^N\).

**Proof.** Suppose that \(S\) is a generalized scaling set (of order \(L\)) associated with the dilation \(A \in M_N(\mathbb{R})\). Then (II.22) holds by the definition of generalized scaling set. Furthermore, by Proposition II.4.1 there exists a wavelet set \(W\) (of order \(L\)) associated with \(A\) such that \(S = \bigcup_{j=1}^\infty (A^T)^{-j}W\), yielding the following:

\[
S = \bigcup_{j=1}^\infty (A^T)^{-j}W \subset \bigcup_{j=0}^\infty (A^T)^{-j}W = A^T S.
\]

Since the \((A^T)^{-j}W\) are pairwise disjoint with respect to \(j\), this also yields

\[
\lim_{n \to \infty} \mathbb{1}_S((A^T)^{-n}\xi) = \lim_{n \to \infty} \sum_{j=-n+1}^{\infty} \mathbb{1}_W((A^T)^j\xi) = \sum_{j \in \mathbb{Z}} \mathbb{1}_W((A^T)^j\xi) = 1
\]

by (II.4). Finally, to obtain (II.24) we perform the calculation
\[
\sum_{\omega \in \Omega} \sum_{k \in \mathbb{Z}^N} \mathbb{1}_S(\xi + (A^T)^{-1}\omega + k) \\
= \sum_{\omega \in \Omega} \sum_{k \in \mathbb{Z}^N} \sum_{j=1}^{\infty} \mathbb{1}_W((A^T)^j(\xi + (A^T)^{-1}\omega + k)) \\
= \sum_{\gamma \in \Gamma} \sum_{j=0}^{\infty} \mathbb{1}_W((A^T)^j(\xi^T \gamma + \omega^T k)) \\
= \sum_{\omega' \in \Omega'} \sum_{k \in \mathbb{Z}^N} \sum_{j=0}^{\infty} \mathbb{1}_W((A^T)^j(\xi^T \omega' + \omega^T + k)) \\
= \sum_{\omega' \in \Omega'} \sum_{k \in \mathbb{Z}^N} \mathbb{1}_W(\xi^T \omega' + \omega^T + k) + \sum_{\omega \in \Omega} \sum_{k \in \mathbb{Z}^N} \sum_{j=1}^{\infty} \mathbb{1}_W((A^T)^j(\xi^T \omega' + \omega^T + k)) \\
= |\Omega'| \cdot L + \sum_{\omega' \in \Omega'} \sum_{k \in \mathbb{Z}^N} \mathbb{1}_S(\xi^T \omega' + \omega^T + k) \\
= \sum_{j=1}^{\infty} \mathbb{1}_W((A^T)^j(\xi^T + \omega' + \omega^T + k)) \\
\text{(II.25)}
\]

where the last equality depends on (II.3) and the fact \( \mathbb{1}_S(\xi) = \sum_{j=1}^{\infty} \mathbb{1}_W((A^T)^j(\xi^T + \omega' + \omega^T + k)) \).
Conversely, suppose that $S$ satisfies (II.21)–(II.24). In light of (II.22), we need only to show that $A^T S \setminus S$ is a wavelet set (of order $L$) associated with $A$. We first claim that

$$
\bigcup_{j=1}^{\infty} \left( (A^T)^{-j}(A^T S \setminus S) \right) = S \mbox{ (modulo null sets)} \quad \mbox{(II.26)}
$$

Indeed, by repeated iterations of (II.21) we have that $(A^T)^j S \subset (A^T)^{j+1} S$ for all $j \in \mathbb{Z}$. Furthermore, $(A^T)^j(A^T S \setminus S) = (A^T)^{j+1} S \setminus (A^T)^j S$ for $j \in \mathbb{Z}$. This produces two important properties:

i) the sets $(A^T)^j(A^T S \setminus S)$ are pairwise disjoint with respect to $j \in \mathbb{Z}$ \quad \mbox{(II.27)}

$$
\bigcup_{j=1}^{\infty} \left( (A^T)^{-j}(A^T S \setminus S) \right) \subset S \quad \mbox{(II.28)}
$$

Now by (II.22) we have

$$
|A^T S \setminus S| = (|\det(A)| - 1)|S| = L \quad \mbox{(II.29)}
$$

and so combining (II.27) and (II.29) we have

$$
\left| \bigcup_{j=1}^{\infty} \left( (A^T)^{-j}(A^T S \setminus S) \right) \right| = \sum_{j=1}^{\infty} \left| (A^T)^{-j}(A^T S \setminus S) \right| = \sum_{j=1}^{\infty} |\det(A)|^{-j}L = \frac{L}{|\det(A)| - 1} = |S| \quad \mbox{(II.30)}
$$
Therefore, by combining (II.28) and (II.30) we obtain (II.26).

This, in turn, yields

\[ 1_S(\xi) = \sum_{j=1}^{\infty} 1_{A^TS \setminus S}( (A^T)^j \xi) \]

and so

\[ \sum_{j \in \mathbb{Z}} 1_{A^TS \setminus S}( (A^T)^j \xi) = \lim_{n \to \infty} 1_S((A^T)^{-n} \xi) = 1 \]

for almost every \( \xi \in \mathbb{R}^N \) by (II.23), showing that \( A^TS \setminus S \) satisfies (II.4).

In addition, (II.26) allows us to repeat the calculation in (II.25) to obtain

\[ \sum_{\omega \in \Omega} \sum_{k \in \mathbb{Z}^N} 1_S(\xi + (A^T)^{-1} + k) = \sum_{\omega' \in \Omega'} \sum_{k \in \mathbb{Z}^N} 1_{A^TS \setminus S}(A^T\xi + \omega' + k) + \sum_{\omega' \in \Omega'} \sum_{k \in \mathbb{Z}^N} 1_S(A^T\xi + \omega' + k) \]

and so (II.24) implies that \( \sum_{k \in \mathbb{Z}^N} 1_{A^TS \setminus S}(\xi + k) = L \) for almost every \( \xi \in \mathbb{R}^N \) and so \( A^TS \setminus S \) satisfies (II.3).
CHAPTER III

DIMENSION FUNCTIONS

III.1 The Dimension Function of a Shift Invariant Space

We begin this chapter with a discussion of shift invariant spaces and dimension functions, as this is the origin of dimension functions of wavelets. A full treatment of this topic was given in De Boor, DeVore, and Ron's work *The Structure of Finitely Generated Shift-Invariant Spaces in L_2(\mathbb{R}^d)* ([17]), and is reiterated very nicely in [6] and [10]. We will summarize the results below.

A subspace \( V \subset L^2(\mathbb{R}^N) \) is called *shift invariant* if for every \( f \in V \) and \( k \in \mathbb{Z}^N \) we have \( T_k f \in V \) (where \( T_k f(x) = f(x - k) \) for each \( x \in \mathbb{R}^N \)).

Given any collection \( \Phi \subset L^2(\mathbb{R}^N) \), we define the *shift invariant system generated by \( \Phi \) to be*

\[
E(\Phi) = \{ T_k \varphi : \varphi \in \Phi; k \in \mathbb{Z}^N \}
\]

and the *shift invariant space generated by \( \Phi \) to be*

\[
S(\Phi) = \text{span} \{ T_k \varphi : \varphi \in \Phi; k \in \mathbb{Z}^N \}.
\]
By a range function we mean any mapping $J$ from $\mathbb{T}^N$ into the collection of closed subspaces of $\ell^2(\mathbb{Z}^N)$. The range function $J$ is said to be measurable if for each $\nu \in \ell^2(\mathbb{Z}^N)$ the function $\xi \mapsto P_J(\xi)\nu$ is a measurable function of $\xi$, where for each $\xi \in \mathbb{T}^N$ we define $P_J(\xi) : \ell^2(\mathbb{Z}^N) \rightarrow J(\xi)$ to be the orthogonal projection of $\ell^2(\mathbb{Z}^N)$ onto $J(\xi)$.

We say that two range functions are equivalent if they agree almost everywhere. Then there is a one-to-one correspondence between the collection of shift invariant spaces and the collection of range functions given by the characterization

$V$ is shift invariant if and only if $V = \{ f \in L^2(\mathbb{R}^N) : \forall f(\xi) \in J_V(\xi) \text{ for a.e. } \xi \in \mathbb{T}^N \}$

for some range function $J_V$, where for each $f \in L^2(\mathbb{R}^N)$ we define

$\forall f : \mathbb{T}^N \rightarrow \ell^2(\mathbb{Z}^N)$ by

$\forall f(\xi) = (\hat{f}(\xi + k))_{k \in \mathbb{Z}^N}.$

In fact, if $V = S(\Phi)$ for some countable collection $\Phi \subset L^2(\mathbb{R}^N)$, then

$J_V(\xi) = \text{span} \{ \forall\varphi(\xi) : \varphi \in \Phi \}.$

Equating range functions that agree almost everywhere makes $J_V$ unique and we refer to it as "the range function of $V". This brings us to the dimension function of a shift invariant space. If $V$ is a shift invariant subspace of $L^2(\mathbb{R}^N)$, then the dimension function of $V$ is the $\mathbb{Z}^N$-periodic function $\dim_V : \mathbb{R}^N \rightarrow \mathbb{N} \cup \{0, \infty\}$ defined by
\[ \dim_V(\xi) = \dim J_V(\xi). \]

For our purposes, however, it will often be easier to calculate \( \dim_V \) by employing a function other than the range function. This alternative function is called the *spectral function* and is defined as follows. The spectral function of a shift invariant space \( V \) is a measurable mapping \( \sigma_V : \mathbb{R}^N \rightarrow [0, 1] \) defined for each \( \xi \in \mathbb{T}^N \) and \( k \in \mathbb{Z}^N \) by

\[ \sigma_V(\xi + k) = \| P_{J_V}(\xi)e_k \|^2, \]

where \( (e_k)_{k \in \mathbb{Z}^N} \) is the standard basis of \( \ell^2(\mathbb{Z}^N) \). A useful property of the spectral function that is not obvious is the fact that

\[ \sum_{k \in \mathbb{Z}^N} \sigma_V(\xi + k) = \dim_V(\xi). \]

To see this, it is enough to notice that

\[ \sum_{k \in \mathbb{Z}^N} \sigma_V(\xi + k) = \sum_{k \in \mathbb{Z}^N} \| P_{J_V}(\xi)e_k \|^2 = \dim \left( \text{ran} \left( P_{J_V}(\xi) \right) \right) = \dim J_V(\xi) = \dim V(\xi). \]

Moreover, under certain conditions on \( V \) we have a very user friendly formula for the spectral function and, hence, the dimension function of \( V \). To state these conditions, it must be clear what is meant be the term *Parseval frame*. A collection of elements \( \mathcal{F} \) in a Hilbert space \( \mathbb{H} \) is a Parseval frame (or tight frame with constant 1) for \( \mathbb{H} \) if for all \( x \in \mathbb{H} \) we have
\[ \|x\|^2 = \sum_{f \in \mathcal{F}} |\langle x, f \rangle|^2. \]

Combining the preceding ideas yields a very succinct formula for \( \dim_V \) under certain circumstances.

**Lemma III.1.1.** If \( V = S(\Phi) \) for some countable \( \Phi \subset L^2(\mathbb{R}^N) \) and \( E(\Phi) \) is a Parseval frame for \( S(\Phi) \), then for each \( \xi \in \mathbb{R}^N \) we have

\[ \dim_V(\xi) = \sum_{k \in \mathbb{Z}^N} \sum_{\varphi \in \Phi} |\hat{\varphi}(\xi + k)|^2. \quad (III.1) \]

**Proof.** Let \( \Phi \) be a countable subset of \( L^2(\mathbb{R}^N) \), and suppose that \( V = S(\Phi) \) and \( E(\Phi) \) is a Parseval frame for \( V \). Then for almost every \( \xi \in \mathbb{R}^N \) we have that \( \{ \mathcal{V} \varphi(\xi) : \varphi \in \Phi \} \) is a Parseval frame for \( J_V(\xi) \) for almost every \( \xi \in \mathbb{T}^N \) (see [6], Theorem 2.5). Using this along with the fact that projections are self-adjoint, we have

\[ \|P_J(\xi)x\|^2 = \sum_{\varphi \in \Phi} |\langle P_J(\xi)x, \mathcal{V} \varphi(\xi) \rangle|^2 = \sum_{\varphi \in \Phi} |\langle x, P_J(\xi)\mathcal{V} \varphi(\xi) \rangle|^2 = \sum_{\varphi \in \Phi} |\langle x, \mathcal{V} \varphi(\xi) \rangle|^2 \]

for every \( x \in \ell^2(\mathbb{Z}^N) \) and almost every \( \xi \in \mathbb{T}^N \). Thus, for each \( k \in \mathbb{Z}^N \) we have

\[ \sigma_V(\xi + k) = \|P_J(\xi)e_k\|^2 = \sum_{\varphi \in \Phi} |\langle e_k, \mathcal{V} \varphi(\xi) \rangle|^2 = \sum_{\varphi \in \Phi} |\hat{\varphi}(\xi + k)|^2, \]
where \((e_k)_{k \in \mathbb{Z}^N}\) is the standard basis of \(l^2(\mathbb{Z}^N)\), and therefore

\[
\dim \nu(\xi) = \sum_{k \in \mathbb{Z}^N} \sigma \nu(\xi + k) = \sum_{k \in \mathbb{Z}^N} \sum_{\varphi \in \Phi} |\hat{\varphi}(\xi + k)|^2.
\]

III.2 Dimension Functions of Integer Dilated Wavelets

Let \(\Psi = \{\psi^1, \ldots, \psi^L\} \subseteq L^2(\mathbb{R}^N)\) be an integer dilated wavelet. That is, \(\Psi\) is a wavelet associated with some dilation \(A \in M_N(\mathbb{Z})\). We begin by giving the definition of the dimension function of such a wavelet. A priori, it has nothing to do with the dimension function of a shift invariant space. It is defined independently as follows:

**Definition III.2.1.** If \(\Psi = \{\psi^1, \ldots, \psi^L\} \subseteq L^2(\mathbb{R}^N)\) is a wavelet associated with \(A \in M_N(\mathbb{Z})\), then the dimension function of \(\Psi\) is the \(\mathbb{Z}^N\)-periodic function \(D_\Psi : \mathbb{R}^N \to \mathbb{N} \cup \{0\}\) given by

\[
D_\Psi(\xi) = \sum_{\ell=1}^L \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^N} |\hat{\psi}^j((A^\ell)^j(\xi + k))|^2.
\]

The fact that \(D_\Psi\) is \(\mathbb{Z}^N\)-periodic is clear. The fact that it is finite (almost everywhere) is not. However, it will follow from the proof of Theorem III.3.8 when we show that \(\int_{\mathbb{R}^N} D_\Psi(\xi) d\xi < \infty\) (and, hence, \(D_\Psi(\xi) < \infty\) for almost every \(\xi \in \mathbb{R}^N\)).
What is also not obvious is why it is called the "dimension function" of $\Psi$. We give a short explanation for this.

Recall that the core space of $\Psi$ is defined as

$$V_0 = \text{span} \{\psi_{j,k}^\ell : j < 0; k \in \mathbb{Z}^N; \ell = 1, \ldots, L\}.$$ 

It can be shown that $V_0$ is a shift invariant subspace of $L^2(\mathbb{R}^N)$ when $\Psi$ is an integer dilated wavelet. We omit proof of this fact as it will follow trivially from a proposition to be proven in Section 3.3. As a shift invariant space, we can define a dimension function for $V_0$ as described in the preceding section. This produces the $\mathbb{Z}^N$-periodic function $\dim_{V_0} : \mathbb{R}^N \rightarrow \mathbb{N} \cup \{0, \infty\}$. We will show that

$$\mathcal{D}_\Psi(\xi) = \dim_{V_0}(\xi)$$

for all $\xi \in \mathbb{R}^N$. Hence the name "dimension function". This fact will follow from results in the next section when we prove that it is true for $A \in M_N(\mathbb{Q})$. However, in the case of $A \in M_N(\mathbb{Z})$ it is interesting to note that the name "dimension function" can be justified directly without the use of range and spectral functions. We do this by defining for each $j \geq 1$ and $\ell = 1, \ldots, L$ and each $\xi \in \mathbb{T}^N$ the vector $\nu_j^\ell(\xi) \in \ell^2(\mathbb{Z}^N)$ by

$$\nu_j^\ell(\xi) = \left(\overline{\psi^\ell((A^T)^j(\xi + k))}\right)_{k \in \mathbb{Z}^N}.$$ 

It can be shown that for almost every $\xi \in \mathbb{T}^N$ we have the following two properties
\[
\sum_{\ell=1}^{L} \sum_{j=1}^{\infty} \|\nu_j^\ell(\xi)\|_{L^2(\mathbb{Z}^N)}^2 = \mathcal{D}_{\psi}(\xi) < \infty
\]

\[
\nu_j^\ell(\xi) = \sum_{\ell'=1}^{L} \sum_{j'=1}^{\infty} \langle \nu_j^\ell(\xi), \nu_{j'}^{\ell'}(\xi) \rangle_{L^2(\mathbb{Z}^N)} \nu_{j'}^{\ell'}(\xi) \quad \text{for every} \quad j \geq 1; \ell = 1, \ldots, L.
\]

We then make use of the following fact about Hilbert spaces. It is sometimes called \textit{Auscher's geometric lemma} and appears in its most general form in [1], Proposition 5.2. A form more closely resembling the one below can be found in [18], Chapter 7, Lemma 3.7.

\textbf{Lemma III.2.2.} If $\mathbb{H}$ is a Hilbert space and $\{x_n : n \geq 1\} \subset \mathbb{H}$ is a collection of vectors satisfying the following two conditions

\[
\sum_{n=1}^{\infty} \|x_n\|^2 < \infty \tag{III.2}
\]

\[
x_n = \sum_{m=1}^{\infty} \langle x_n, x_m \rangle x_m \quad \text{for all} \quad n \geq 1 \tag{III.3}
\]

then \(\dim \text{span} \{x_n : n \geq 1\} = \sum_{n=1}^{\infty} \|x_n\|^2.\)

\textit{Proof.} Define the linear operator $\mathcal{J}$ on $\mathbb{H}$ by

\[
\mathcal{J} x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n.
\]
This is well-defined by the following argument: For any $M, N \geq 1$ with $M < N$ we have

$$\left\| \sum_{n=M}^{N} \langle x, x_n \rangle x_n \right\| \leq \sum_{n=M}^{N} \| \langle x, x_n \rangle x_n \| = \sum_{n=M}^{N} \| \langle x, x_n \rangle \| \| x_n \| \leq \sum_{n=M}^{N} \| x \| \| x_n \| ^2 = \| x \| \sum_{n=M}^{N} \| x_n \| ^2. \quad \text{(III.4)}$$

But (III.2) and the Cauchy criterion for convergence imply that

$$\lim_{M \to \infty} \sum_{n=m}^{N} \| x_n \| ^2 = 0.$$ Thus, by (III.4), $\lim_{M \to \infty} \left\| \sum_{n=M}^{N} \langle x, x_n \rangle x_n \right\| = 0$ and, hence, $\sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$ converges in $\mathbb{H}$ by the Cauchy criterion.

In addition to being well-defined, $J$ is also bounded (and, hence, continuous) since for all $x \in \mathbb{H}$ we have by (III.4)

$$\left\| Jx \right\| \leq \| x \| \sum_{n=1}^{\infty} \| x_n \| ^2.$$

Next, we claim that $\overline{\text{span}} \ \{ x_n : n \geq 1 \} = \ker(J - J)$ where $J$ is the identity operator on $\mathbb{H}$. Indeed, by (III.3) we have $Jx_n - x_n = 0$ for all $n \geq 1$. Hence, since $J$ is continuous, $Jx - x = 0$ for all $x \in \overline{\text{span}} \ \{ x_n : n \geq 1 \}$, giving $x \in \overline{\text{span}} \ \{ x_n : n \geq 1 \} \subset \ker(J - J)$. Furthermore, by linearity of $J$, we have $\ker(J - J) \subset \text{ran}(J)$ since $x \in \ker(J - J)$ implies that $Jx - x = 0$ and so $x = Jx$ (showing that $x \in \text{ran}(J)$). Finally, $\text{ran}(J) \subset \overline{\text{span}} \ \{ x_n : n \geq 1 \}$ by the definition of $J$. Therefore, $\overline{\text{span}} \ \{ x_n : n \geq 1 \} \subset \ker(J - J) \subset \text{ran}(J) \subset \overline{\text{span}} \ \{ x_n : n \geq 1 \}$.

Now let $\{ e_k \}_{k \in \mathcal{K}}$ be an orthonormal basis of $\overline{\text{span}} \ \{ x_n : n \geq 1 \}$. Then $e_k \in \ker(J - J)$ for each $k \in \mathcal{K}$ and so we have
\[ |\mathcal{K}| = \sum_{k \in \mathcal{K}} \langle e_k, e_k \rangle = \sum_{k \in \mathcal{K}} \langle \mathcal{J}e_k, e_k \rangle = \sum_{k \in \mathcal{K}} \left\langle \sum_{n=1}^{\infty} \langle e_k, x_n \rangle x_n, e_k \right\rangle = \sum_{n=1}^{\infty} \sum_{k \in \mathcal{K}} |\langle x_n, e_k \rangle|^2 = \sum_{n=1}^{\infty} \|x_n\|^2. \]

Thus \( \dim \text{span} \{ x_n : n \geq 1 \} = \sum_{n=1}^{\infty} \|x_n\|^2. \)

Applying this result to the vectors \( \{ \nu_j^\ell(\xi) : j \geq 1; \ell = 1, \ldots, L \} \) we get

\[ \mathcal{D}_\Psi(\xi) = \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} \|\nu_j^\ell(\xi)\|^2_{B(L^2)} = \dim \text{span} \{ \nu_j^\ell(\xi) : j \geq 1; \ell = 1, \ldots, L \}. \]

A central result in the theory of dimension functions of integer dilated wavelets is the characterization given by Bownik, Rzeszotnik, and Speegle in [11], Theorem 4.2, and independently by Baggett and Merrill in [4], which reads as follows:

**Theorem III.2.3.** A \( \mathbb{Z}^N \)-periodic function \( \mathcal{D} : \mathbb{R}^N \to \mathbb{N} \cup \{0\} \) is the dimension function of a wavelet \( \Psi = \{ \psi^1, \ldots, \psi^L \} \) associated with the dilation \( A \in M_N(\mathbb{Z}) \) if and only if the following conditions are satisfied:
\[
\int_{\mathbb{T}^N} \mathcal{D}(\xi) d\xi = \frac{L}{|\det(A)| - 1}
\]

\[
\liminf_{n \to \infty} \mathcal{D}((A^T)^{-n} \xi) \geq 1 \text{ for a.e. } \xi \in \mathbb{R}^N
\]

\[
\sum_{\lambda \in \Lambda} \mathcal{D}(\xi + (A^T)^{-1} \lambda) = L + \mathcal{D}(A^T \xi) \text{ for a.e. } \xi \in \mathbb{R}^N \tag{III.5}
\]

\[
\sum_{k \in \mathbb{Z}^N} \mathbf{1}_\Delta(\xi + k) \geq \mathcal{D}(\xi) \text{ for a.e. } \xi \in \mathbb{R}^N
\]

where $\Delta = \{ \xi \in \mathbb{R}^N : \mathcal{D}((A^T)^{-j} \xi) \geq 1 \text{ for all } j \in \mathbb{N} \cup \{0\} \}$ and $\Lambda$ is a transversal of $\mathbb{Z}^N / A^T \mathbb{Z}^N$.

Again, we omit proof of the necessity of these four conditions as it will follow from a more general statement in Section 3.3. However, we will take a moment to discuss how the sufficiency of these four conditions is obtained. Bownik, Rzeszotnik, and Speegle offer the following algorithm:

**Remark III.2.4.** In the following algorithm, we let $A_k = \{ \xi \in \mathbb{T}^N : \mathcal{D}(\xi) \geq k \}$ and $X^p = \bigcup_{k \in \mathbb{Z}^N} \{ x + k : x \in X \}$ for any set $X \subset \mathbb{R}^N$. Furthermore, given $\tilde{X} \subset \mathbb{R}^N$ we use $X$ to denote any subset of $\tilde{X}$ on which the translation projection $\tau$ is injective and has image $\tau(\tilde{X})$, whose existence is guaranteed by Claim II.2.3.
Algorithm III.2.5. Assume that $\mathcal{D}$ satisfies the conditions set forth in Theorem III.2.3 for some $L$ and $A \in M_N(Z)$, then perform the following construction:

1. Let $E_1$ be any measurable subset of $\mathbb{R}^N$ satisfying the following four properties:
   
   i. $E_1 \subset A^T E_1$
   
   ii. $\lim_{n \to \infty} \mathbb{I}_{E_1}((A^T)^{-n} \xi) = 1$ for almost every $\xi \in \mathbb{R}^N$
   
   iii. $\tau|_{E_1}$ is injective
   
   iv. $\mathcal{D}(\xi) \geq 1$ for all $\xi \in E_1$.

   Then, for each $n \in \mathbb{N}$ let $\tilde{E}_{n+1} = (A^T E_n \setminus \bigcup_{i=1}^{n} E_i^P) \cap A_1^P$ and define $S_1 = \bigcup_{n=1}^{\infty} E_n$.

2. Suppose that $S_i$ is defined for all $i = 1, \ldots, m - 1$ and let $P_{m-1} = \bigcup_{i=1}^{m-1} S_i$.

3. Let $\tilde{F}_{m,1} = (A^T P_{m-1} \setminus P_{m-1}) \cap A_m^P$. Then, for each $n \in \mathbb{N}$ let $\tilde{F}_{m,n+1} = (A^T F_{m,n} \setminus \bigcup_{i=1}^{n} F_{m,i}^P) \cap A_m^P$ and define $S_m = \bigcup_{n=1}^{\infty} F_{m,n}$.

Bowen, Rzeszotnik, and Speegle prove (see [11], pages 80-83) that constructing $S_n$ for each $n \in \mathbb{N}$ as in Algorithm III.2.5 and defining $S$ to be $S = \bigcup_{n \in \mathbb{N}} S_n$ produces a set with the following qualities:
\[ S \subset A^T S \]  
(III.6)

\[ |S| = \frac{L}{|\det(A)| - 1} \]  
(III.7)

\[ \lim_{n \to \infty} 1_S((A^T)^{-n}\xi) = 1 \text{ for a.e. } \xi \in \mathbb{R}^N \]  
(III.8)

\[ \mathcal{D}(\xi) = \sum_{k \in \mathbb{Z}^N} 1_S(\xi + k). \]  
(III.9)

Notice that (III.6)–(III.8) are exactly (II.21)–(II.23). Furthermore, close inspection shows that (III.9) in conjunction with (III.5) yields (II.24). Indeed, since \( A \in M_N(\mathbb{Z}) \), then \( A^T \mathbb{Z}^N \subset \mathbb{Z}^N \) and, hence, \( \Omega = \mathbb{Z}^N / A^T \mathbb{Z}^n = \Lambda \) and 
\[ \Omega' = \mathbb{Z}^N / \mathbb{Z}^N = \{0\} \]. Therefore, by Theorem II.4.2, \( S \) is a generalized scaling set (of order \( L \)) associated with \( A \).

Once we have \( S \), the proof of Theorem II.4.2 shows that defining \( W \) to be 
\( A^T S \setminus S \) gives a wavelet set (of order \( L \)) associated with \( A \). Then defining 
\( W_1, \ldots, W_L \) as in the proof of Theorem II.2.1 and defining \( \Psi = \{\psi^1, \ldots, \psi^L\} \) by 
\( \hat{\psi}^\ell = 1_{W_\ell} \) for each \( \ell = 1, \ldots, L \) produces an MSF wavelet associated with \( A \).

Furthermore, if we calculate the dimension function of this wavelet, we get

\[ \mathcal{D}_\Psi(\xi) = \sum_{\ell=1}^L \sum_{j=1}^\infty \sum_{k \in \mathbb{Z}^N} |\hat{\psi}^\ell((A^\ell)(\xi + k))|^2 = \sum_{\ell=1}^L \sum_{j=1}^\infty \sum_{k \in \mathbb{Z}^N} 1_{W_\ell}((A^\ell)(\xi + k)) \]
\[ = \sum_{k \in \mathbb{Z}^N} 1_S(\xi + k) = \mathcal{D}(\xi), \quad \text{(III.10)} \]

showing that \( \mathcal{D} \) is the dimension function of \( \Psi \).

We would like to extend this characterization to rationally dilated wavelets. A priori, it is not obvious what modifications will be necessary. However, it is clear that (III.5) will have to be reformulated, if only because \( \Lambda \) is not defined when \( A^\top \mathbb{Z}^N \not\subseteq \mathbb{Z}^N \). This condition is commonly known as the consistency condition and it appears frequently in the study of wavelets. It was first introduced by Baggett, Medina, and Merrill in conjunction with a construction known as a generalized multi-resolution analysis (see [3]). One can be certain that it will be essential in any characterization of dimension functions of rationally dilated wavelets. However, it will have to be written in a more general form to account for the possibility that \( A^\top \mathbb{Z}^N \not\subseteq \mathbb{Z}^N \). We will see how to overcome this in the next section.

III.3 Dimension Functions of Rationally Dilated Wavelets

We wish to define the dimension function of a rationally dilated wavelet. With that in mind, we suggest the following definition.

**Definition III.3.1.** Let \( \Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^N) \) be a wavelet associated with the dilation \( \Lambda \in M_N(\mathbb{Q}) \). The dimension function of \( \Psi \) is the function
\[ \mathcal{D}_\Psi : \mathbb{R}^N \rightarrow \mathbb{N} \cup \{0\} \text{ given by} \]

\[ \mathcal{D}_\Psi (\xi) = \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^N} \left| \hat{\psi}_j^\ell ((A_\ell)^j (\xi + k)) \right|^2. \]  

(III.11)

Notice that this definition is identical to that of an integer dilated wavelet. This is a testament to the robustness of that definition, as it is not dependent upon \( A \) being an integer dilation. However, stating a definition and demonstrating that it is appropriate are two altogether different things. We will now show that Definition III.3.1 makes sense as stated by showing that

\[ \mathcal{D}_\Psi (\xi) = \dim_{\mathcal{V}_0} (\xi) \]

for every \( \xi \in \mathbb{R}^N \) when \( \mathcal{V}_0 \) is shift invariant. Notice the intentional use of the word \textit{when}. Indeed, it is not known whether the core space of a rationally dilated wavelet must, by necessity, be shift invariant. Thus, it is desirable to have some confirmation that it may, in fact, be so. To this aim, we now spend some time discussing conditions on \( A \in M_N(\mathbb{R}) \) that guarantee the shift invariance of \( \mathcal{V}_0 \). Let us begin with the following lemma.

**Lemma III.3.2.** Suppose \( \Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^N) \) is a wavelet associated with the dilation \( A \in M_N(\mathbb{R}) \). Then \( \mathcal{V}_0 \) is shift invariant if

\[ T_k \psi^\ell_{h,m} \perp \psi^\ell_{j,n} \]  

(III.12)

for all \( h < 0; j \geq 0; k, m, n \in \mathbb{Z}^N; \ell, \ell' = 1, \ldots, L \).
Proof. Suppose that (III.12) holds. Then we can go one step further and say that $T_{k_1} \psi^\ell_{h,m} \perp T_{k_2} \psi^\ell_{j,n}$ for all $h < 0$; $j \geq 0$; $k_1, k_2, m, n \in \mathbb{Z}^N$; $\ell, \ell' = 1, \ldots, L$. Indeed, by using the change of variables $x - k_2 \mapsto x$ we have

$$\langle T_{k_1} \psi^\ell_{h,m}, T_{k_2} \psi^\ell_{j,n} \rangle = \langle T_{k_1 - k_2} \psi^\ell_{h,m}, \psi^\ell_{j,n} \rangle = 0.$$ 

Now define the following:

$$W_1 = \overline{\text{span}} \{ T_k \psi^\ell_{j,n} : j < 0; k, n \in \mathbb{Z}^N; \ell = 1, \ldots, L \}$$

$$W_2 = \overline{\text{span}} \{ T_k \psi^\ell_{j,n} : j \geq 0; k, n \in \mathbb{Z}^N; \ell = 1, \ldots, L \}.$$ 

Recall that $V_0 = \overline{\text{span}} \{ \psi^\ell_{j,n} : j < 0; n \in \mathbb{Z}^N; \ell = 1, \ldots, L \}$ and, hence, $V_0^\perp = \overline{\text{span}} \{ \psi^\ell_{j,n} : j \geq 0; n \in \mathbb{Z}^N; \ell = 1, \ldots, L \}$. Thus, we have $V_0 \subset W_1$ and $V_0^\perp \subset W_2$. Furthermore, since $\Psi$ is a wavelet then $V_0 \oplus V_0^\perp = L^2(\mathbb{R}^N)$ and, hence, $W_1 + W_2 = L^2(\mathbb{R}^N)$. Thus, $W_1 + W_2 = V_0 \oplus V_0^\perp$. Therefore, since $V_0 \subset W_1$ and $V_0^\perp \subset W_2$, we have $W_1 = V_0$ and $W_2 = V_0^\perp$. Since $W_1$ is, by definition, shift invariant, then so is $V_0$.

$\square$

**Proposition III.3.3.** Suppose $\Psi = \{ \psi^1, \ldots, \psi^L \} \subset L^2(\mathbb{R}^N)$ is a wavelet associated with the dilation $A \in M_N(\mathbb{Q})$. If $A$ is diagonal, then $V_0$ is shift invariant.
Proof. Suppose \( A = \text{diag}(\frac{p_1}{q_1}, \ldots, \frac{p_N}{q_N}) \) where \( p_i, q_i \in \mathbb{Z} \) with \( \gcd(p_i, q_i) = 1 \) for each \( i = 1, \ldots, N \) and \( I_{N 	imes N} \) is the identity in \( M_N(\mathbb{R}) \). Given any \( k = (k_1, \ldots, k_N) \in \mathbb{Z}^N \) and any \( h < 0 \) and \( j \geq 0 \), let \( (\omega_1, \ldots, \omega_N) \in \mathbb{Z}^N \) where \( \omega_i \) satisfies the two following conditions for each \( i = 1, \ldots, N \):

\[
\omega_i \equiv k_i \pmod{p_i^{-h}} \\
\omega_i \equiv 0 \pmod{q_i^j}.
\]

Then for each \( i = 1, \ldots, N \) we have \( (\frac{p_i}{q_i})^h(\omega_i - k_i) \in \mathbb{Z} \) and \( (\frac{p_i}{q_i})^j \omega_i \in \mathbb{Z} \).

Hence, \( A^h(\omega - k) \in \mathbb{Z}^N \) and \( A^j \omega \in \mathbb{Z}^N \). Thus, for each \( m, n \in \mathbb{Z}^N \) and \( \ell, \ell' = 1, \ldots, L \) we have

\[
\langle T_{k \psi_{h,m}^\ell}, \psi_{j,n}^{\ell'} \rangle = \langle T_{kD_hT_m \psi^\ell}, D_jT_n \psi^{\ell'} \rangle = \langle T_{\omega+k}D_hT_m \psi^\ell, T_\omega D_j T_n \psi^{\ell'} \rangle
\]

\[
= \langle D_hT_A^{h(\omega+k)+m} \psi^\ell, D_jT_A^{\omega+n} \psi^{\ell'} \rangle = \langle \psi_{h,-A^{h(\omega+k)+m}}^{\ell}, \psi_{j,-A^{\omega+n}}^{\ell'} \rangle = 0.
\]

Therefore, \( V_0 \) is shift invariant by Lemma III.3.2.

The preceding lemma and proposition are essentially those used by Bownik and Speegle in investigating dimension functions of wavelets in \( L^2(\mathbb{R}^1) \) (see [12], Theorem 3.2). This is sufficient when \( N = 1 \) and it shows that the core space of

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every rationally dilated wavelet in $L^2(\mathbb{R})$ is shift invariant. However, when $N > 1$ it is lacking in the sense that it only addresses a relatively small collection of possible dilations. Thus, we would like to have a weaker condition on $A$ that implies shift invariance of the core space. Fortunately, there is such a condition. Furthermore, it does not require that $A$ be rational.

**Theorem III.3.4.** Suppose $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^N)$ is a wavelet associated with the dilation $A \in M_N(\mathbb{R})$. Then $V_0$ is shift invariant if for all $h < 0$ and $j \geq 0$ we have the following:

$$\mathbb{Z}^N \subset A^h\mathbb{Z}^N + A^j\mathbb{Z}^N.$$  \hfill (III.13)

**Proof.** Take any $h < 0; j \geq 0; k, m, n \in \mathbb{Z}^N; \ell, \ell' = 1, \ldots, L$. If $j = 0$, then

$$\langle T_k\psi^\ell_m, \psi^{\ell'}_{j,n} \rangle = \langle T_k\psi^\ell_m, D_jT_n\psi^{\ell'} \rangle = \langle \psi^\ell_m, T_{-k}D_jT_n\psi^{\ell'} \rangle$$

$$= \langle \psi^\ell_m, D_jT_{-k}\psi^{\ell'} \rangle = \langle \psi^\ell_m, \psi^{\ell'}_{j,n-k} \rangle = 0.$$

If, on the other hand, $j > 0$ then we choose $\alpha, \beta \in \mathbb{Z}^N$ such that $k = -A^{-h}\alpha - A^{-j}\beta$. We are guaranteed the existence of such $\alpha, \beta \in \mathbb{Z}^N$ by (III.13). Then
\[ \langle T_h \psi^\ell_{h,m}, \psi^\ell_{j,n} \rangle = \langle T_{-A^{h\alpha} - A^{-1} B} D_h T_m \psi^\ell, D_j T_n \psi^\ell \rangle \]
\[ = \langle D_h T_{-A^{h\alpha}} \psi^\ell, D_j T_{n+B} \psi^\ell \rangle = \langle \psi^\ell_{h,m-A^{h\alpha}}, \psi^\ell_{j,n+B} \rangle = 0. \]

Therefore, \( V_0 \) is shift invariant by Lemma III.3.2.

\[ \square \]

It is important to understand, however, that Theorem III.3.4 does not guarantee the shift invariance of the core space of every rationally dilated wavelet, as we show with the following example:\(^1\)

**Example III.3.5.** If \( A = \begin{pmatrix} 1 & 2 \\ 3/2 & -1 \end{pmatrix} \), then \( A \) is an expansive matrix that does not satisfy (III.13) for \( h = -1 \) and \( j = 1 \).

**Proof.** To see that \( A \) is expansive, note that the eigenvalues are \(-2, 2\). We will show, however, that \( \mathbb{Z}^2 \not\subset A^{-1} \mathbb{Z}^2 + A \mathbb{Z}^2 \) by demonstrating that \( (1, 0) \not\in A^{-1} \mathbb{Z}^2 + A \mathbb{Z}^2 \). Indeed, suppose there were some \( \alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{Z}^2 \) such that \( (1, 0) = A^{-1} \alpha + A \beta \). Since \( A^{-1} = \begin{pmatrix} 1/4 & 1/2 \\ 3/8 & -1/4 \end{pmatrix} \),

\(^1\)This example was communicated to Prof. Marcin Bownik by Dr. Daniel Chan of the University of New South Wales, Sydney, Australia

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it follows that \( \frac{1}{4} \alpha_1 + \frac{1}{2} \alpha_2 + \beta_1 + 2 \beta_2 = 1 \) and \( \frac{3}{8} \alpha_1 - \frac{1}{4} \alpha_2 + \frac{3}{2} \beta_1 - \beta_2 = 0 \). These two equations, respectively, yield \( 3 \alpha_1 + 12 \beta_1 = 12 - 6 \alpha_2 - 24 \beta_2 \) and
\[
3 \alpha_1 + 12 \beta_1 = 2 \alpha_2 + 8 \beta_2.
\]
Thus, \( 12 - 6 \alpha_2 - 24 \beta_2 = 2 \alpha_2 + 8 \beta_2 \), implying that
\[
8 \beta_2 = 3 - 2 \alpha_2.
\]
Since \( 8 \beta_2 \) is even and \( 3 - 2 \alpha_2 \) is odd, this is an obvious contradiction.

\[ \square \]

It was promised in the preceding section that the shift invariance of the core space of an integer dilated wavelet would follow readily from the discussion of rationally dilated wavelets. Indeed, it is a direct consequence of Theorem III.3.4 as we now demonstrate.

**Corollary III.3.6.** If \( \Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^N) \) is a wavelet associated with the dilation \( A \in M_N(\mathbb{Z}) \), then \( V_0 \) is shift invariant.

**Proof.** Suppose \( h < 0 \) and \( j \geq 0 \). Then for any \( k \in \mathbb{Z}^N \) we have \( A^{-h}k \in \mathbb{Z}^N \) and, therefore, \( k = A^h A^{-h} k \in A^h \mathbb{Z}^N \). Thus, \( \mathbb{Z}^N \subset A^h \mathbb{Z}^N \subset A^h \mathbb{Z}^N + A^j \mathbb{Z}^N \). Hence, \( V_0 \) is shift invariant by Theorem III.3.4.

\[ \square \]

Hopefully at this point the reader is convinced that discussing the shift invariance of core spaces associated with rationally dilated wavelets is not a fool’s errand. Thus, we return to our discussion of the dimension function by showing that if \( V_0 \) is shift invariant, then its dimension function coincides with that of its associated wavelet.
**Proposition III.3.7.** Suppose that \( \Psi \) is a rationally dilated wavelet whose core space \( V_0 \) is shift invariant. Then \( \mathcal{D}_\Psi(\xi) = \dim V_0(\xi) \) for all \( \xi \in \mathbb{R}^N \), where \( \mathcal{D}_\Psi \) is as in Definition III.3.1.

**Proof.** Let \( \mathcal{A}(\Psi) \) denote the affine system associated with \( (\Psi, A) \). Recall that this is defined to be

\[
\mathcal{A}(\Psi) = \{ \psi_{j,k}^\ell : j \in \mathbb{Z}; k \in \mathbb{Z}^N; \ell = 1, \ldots, L \},
\]

where \( \psi_{j,k}^\ell(x) = |\det(A)|^{1/2} \psi^\ell(A^j x + k) \). In the case of \( A \in M_N(\mathbb{Z}) \) we have \( A\mathbb{Z}^N \subset \mathbb{Z}^N \) and the affine system \( \mathcal{A}(\Psi) \) would be sufficient to prove many statements. However, when \( A\mathbb{Z}^N \nsubseteq \mathbb{Z}^N \), as is often the case when \( A \in M_N(\mathbb{Q}) \), we must resort to a more complicated construction called the *quasi-affine system* associated with \( (\Psi, A) \) which was first introduced by Ron and Shen in [22], Section 5, and was developed also by Bownik in [8]. We will denote the quasi-affine system associated with \( (\Psi, A) \) by \( \mathcal{A}^q \). It is defined as follows.

For any countable collection \( \Phi \subset L^2(\mathbb{R}^N) \), we define the *oversampled system* of \( E(\Phi) \) with respect to the rational lattice \( \Gamma \) by

\[
\mathcal{O}^\Gamma(\Phi) = E\left( \bigcup_{\vartheta \in \Theta} \left\{ \frac{1}{|\mathbb{Z}^N / (\mathbb{Z}^N \cap \Gamma)|^{1/2}} T_{\vartheta} \Phi \right\} \right),
\]

where \( \Theta \) is a transversal of \( (\mathbb{Z}^N + \Gamma)/\mathbb{Z}^N \). We define the quasi-affine system associated with \( (\Phi, A) \) by
\[ A^q(\Phi) = \bigcup_{j \in \mathbb{Z}} O^j \mathbb{Z}^N(D_j \Phi) \]
\[ = E \left( \bigcup_{j \in \mathbb{Z}} \bigcup_{\theta \in \Theta_j} \left\{ \frac{1}{|\mathbb{Z}^N/(\mathbb{Z}^N \cap A^{-j} \mathbb{Z}^N)|^{1/2}} T_\theta D_j \Phi \right\} \right), \quad \text{(III.14)} \]

where, in this case, \( \Theta_j \) is a transversal of \((\mathbb{Z}^N + A^{-j} \mathbb{Z}^N)/\mathbb{Z}^N\).

Given the wavelet \( \Psi = \{\psi^1, \ldots, \psi^L\} \) associated with the dilation \( A \in M_N(\mathbb{Q}) \), we define the lattice \( \Gamma_j \) for each \( j \in \mathbb{Z} \) by \( \Gamma_j = \mathbb{Z}^N + A^{-j} \mathbb{Z}^N \) and let \( M_j \in M_N(\mathbb{Q}) \) be such that \( M_j \mathbb{Z}^N = \Gamma_j \). The result is

\[ A^q(\Psi) = \{\tilde{\psi}_{j,k}^\ell : j \in \mathbb{Z}; k \in \mathbb{Z}^N; \ell = 1, \ldots, L\}, \]

where for each \( j \in \mathbb{Z}, k \in \mathbb{Z}^N, \) and \( \ell = 1, \ldots, L \) we define

\[ \tilde{\psi}_{j,k}^\ell(x) = \frac{|\det(A)|^{j/2}}{|\mathbb{Z}^N/(\mathbb{Z}^N \cap A^{-j} \mathbb{Z}^N)|^{1/2}} \psi^\ell(A^j(x + M_j k)). \]

We have, by definition, \( V_0 = \text{span} \{\psi_{j,k}^\ell : j < 0; k \in \mathbb{Z}^N; \ell = 1, \ldots, L\} \). Now define
\[
A_0^q(\Psi) = \bigcup_{j < 0} O^{A^{-j}Z^N}(D_j \Psi)
\]

\[
= E \left( \bigcup_{j < 0} \bigcup_{\theta \in \Theta_j} \left\{ \frac{1}{|Z^N/(Z^N \cap A^{-j}Z^N)|^{1/2}} T_\theta D_j \Psi \right\} \right)
\]

\[
= \{ \tilde{\psi}_{j,k}^\ell : j < 0; k \in \mathbb{Z}^N; \ell = 1, \ldots, \mathcal{L} \}.
\]

We claim that \( V_0 = \overline{\text{span}} \ A_0^q \). Indeed, given any \( k_1 \in \mathbb{Z}^N \) we have \( A^{-j}k_1 \in A^{-j}Z^N \subseteq \mathbb{Z}^N + A^{-j}Z^N = \Gamma_j \) and so there is some \( k_2 \in \mathbb{Z}^N \) such that \( A^j M_j k_2 = k_1 \), implying that \( \tilde{\psi}_{j,k}^\ell = \frac{1}{|Z^N/(Z^N \cap A^{-j}Z^N)|^{1/2}} \tilde{\psi}_{j,k}^\ell \). Thus, \( V_0 \subseteq \overline{\text{span}} \ A_0^q \).

On the other hand, since \( \mathbb{Z}^N = M_j^{-1} \Gamma_j = M_j^{-1}Z^N + M_j^{-1}A^{-j}Z^N \), then there are \( \alpha, \beta \in \mathbb{Z}^N \) such that \( M_j^{-1} \alpha + M_j^{-1}A^{-j} \beta = k_1 \), implying that

\[
\tilde{\psi}_{j,k_1}^\ell = \frac{1}{|Z^N/(Z^N \cap A^{-j}Z^N)|^{1/2}} T_\alpha \tilde{\psi}_{j,\beta}^\ell. \]

Since \( V_0 \) is shift invariant, this yields \( \tilde{\psi}_{j,k_1}^\ell \in V_0 \) and so \( \overline{\text{span}} \ A_0^q \subseteq V_0 \).

Therefore, we have

\[
V_0 = \overline{\text{span}} E \left( \bigcup_{j < 0} \bigcup_{\theta \in \Theta_j} \left\{ \frac{1}{|Z^N/(Z^N \cap A^{-j}Z^N)|^{1/2}} T_\theta D_j \Psi \right\} \right). \quad (III.15)
\]

(Similarly, we can show that \( L^2(\mathbb{R}^N) \ominus V_0 = \overline{\text{span}} (A^q(\Psi) \setminus A_0^q(\Psi)). \))

We claim also that \( A_0^q(\Psi) \) is a Parseval frame for \( V_0 \). Indeed, by the definition of a wavelet, the affine system \( A(\Psi) \) is an orthonormal basis and, hence, a
Parseval frame for $L^2(\mathbb{R}^N)$. While the quasi-affine system $A^q(\Psi)$ does not inherit the property of being an orthonormal basis, it does in fact inherit the Parseval frame property with respect to $L^2(\mathbb{R}^N)$. This is by Theorem 3.4 of [8] which proves that for any $\mathcal{F} = \{f^1, \ldots, f^L\} \subset L^2(\mathbb{R}^N)$ and any rational dilation $A$, the affine system associated with $(\mathcal{F}, A)$ is a Parseval frame for $L^2(\mathbb{R}^N)$ if and only if its quasi-affine counterpart is a Parseval frame for $L^2(\mathbb{R}^N)$. Furthermore, since \( \text{span} \ A_0^q(\Psi) = V_0 \) and \( \text{span} \ (A^q \setminus A_0^q) = L^2(\mathbb{R}^N) \oplus V_0 \), it follows that $A_0^g$ and $A^g \setminus A_0^g$ are Parseval frames for $V_0$ and $L^2(\mathbb{R}^N) \oplus V_0$, respectively.

Since $A_0^g(\Psi)$ is defined to be $E\left( \bigcup_{j < 0} \bigcup_{\theta \in \Theta_j} \left\{ \frac{1}{|Z^N/(Z^N \cap A^{-j}Z^N)|^{1/2}} T_{\theta} D_j \Psi \right\} \right)$, the preceding argument yields the following:

$$E\left( \bigcup_{j < 0} \bigcup_{\theta \in \Theta_j} \left\{ \frac{1}{|Z^N/(Z^N \cap A^{-j}Z^N)|^{1/2}} T_{\theta} D_j \Psi \right\} \right) \text{ is a Parseval frame for } V_0. \quad (III.16)$$

Thus, by (III.15), (III.16), and (III.1) we have

$$\dim V_0(\xi) = \sum_{\ell=1}^{L} \sum_{j < 0} \sum_{k \in Z^N} \sum_{\theta \in \Theta_j} \left| \frac{1}{|Z^N/(Z^N \cap A^{-j}Z^N)|^{1/2}} T_{\theta} \frac{1}{|Z^N/(Z^N \cap A^{-j}Z^N)|^{1/2}} D_j \Psi \right|^2. \quad (III.17)$$

However, if we employ the fact that

$$\frac{|Z^N/(Z^N \cap A^{-j}Z^N)|}{|A^{-j}Z^N/(Z^N \cap A^{-j}Z^N)|} = |Z^N/A^{-j}Z^N| = |\det(A)|^{-j}$$

(see [26], Proposition 5.5) and the fact that $(Z^N + A^{-j}Z^N)/Z^N$ is isomorphic to $A^{-j}Z^N/(Z^N \cap A^{-j}Z^N)$, then we conclude $|\Theta_j| = |Z^N/(Z^N \cap A^{-j}Z^N)| \cdot |\det(A)|^j$. This, along with the fact that

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\[
\overline{\psi_{j, \theta}^\ell}(\xi + k) = \frac{|\text{det}(A)|^{j/2}}{|Z^N/(Z^N \cap A^{-j}Z^N)|^{1/2}} \overline{\psi^\ell((A^T)^{-j}(\xi + k))} e^{-2\pi i \langle \xi, M_j \theta \rangle},
\]

allows for the simplification

\[
\sum_{\theta \in \Theta_j} |\overline{\psi_{j, \theta}^\ell}(\xi + k)|^2 = \sum_{\theta \in \Theta_j} \left| \frac{|\text{det}(A)|^{j/2}}{|Z^N/(Z^N \cap A^{-j}Z^N)|^{1/2}} \overline{\psi^\ell((A^T)^{-j}(\xi + k))} \right|^2
\]

\[
= \sum_{\theta \in \Theta_j} \frac{1}{|Z^N/(Z^N \cap A^{-j}Z^N)||\text{det}(A)|^j} \left| \overline{\psi^\ell((A^T)^{-j}(\xi + k))} \right|^2
\]

\[
= \frac{|\Theta_j|}{|Z^N/(Z^N \cap A^{-j}Z^N)||\text{det}(A)|^j} \left| \overline{\psi^\ell((A^T)^{-j}(\xi + k))} \right|^2
\]

\[
= \left| \overline{\psi^\ell((A^T)^{-j}(\xi + k))} \right|^2.
\]

Therefore, (III.17) becomes

\[
\dim_{\psi}(\xi) = \sum_{\ell=1}^{L} \sum_{j<0} \sum_{k \in Z^N} \left| \overline{\psi^\ell((A^T)^{-j}(\xi + k))} \right|^2 = \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in Z^N} \left| \overline{\psi^\ell((A^T)^{j}(\xi + k))} \right|^2.
\]

This demonstrates the appropriateness of the term “dimension function” by showing that \( D_\psi \) actually is a dimension function in the sense of shift invariant
spaces. However, it does not attest to the fact that the range of $\mathcal{D}_\Psi$ is $\mathbb{N} \cup \{0\}$.

Recall that in general for a shift invariant space $V$ we have $\dim_V : \mathbb{R}^N \rightarrow \mathbb{N} \cup \{0, \infty\}$. However, the fact that $\mathcal{D}_\Psi(\xi) < \infty$ for (almost) every $\xi \in \mathbb{R}^N$ follows from (III.18) in Theorem III.3.8 below.

The original goal when first undertaking this project was to formulate the definition of the dimension function of a rationally dilated wavelet and provide a characterization for it similar to the case of an integer dilated wavelet. Although a complete characterization remains elusive, we will show that the necessary (and sufficient) conditions for a function to be the dimension function of an integer dilated wavelet can be suitably generalized to give necessary conditions for a function to be the dimension function of a rationally dilated wavelet.

**Theorem III.3.8.** If $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^N)$ is a wavelet associated with the dilation $A \in M_N(\mathbb{Q})$, then the dimension function $\mathcal{D}_\Psi : \mathbb{R}^N \rightarrow \mathbb{N} \cup \{0\}$ satisfies the following four conditions:

\[
\int_{\mathbb{T}^N} \mathcal{D}_\Psi(\xi) d\xi = \frac{L}{|\det(A)| - 1} \tag{III.18}
\]

\[
\liminf_{n \to \infty} \mathcal{D}_\Psi((A^t)^{-n}\xi) \geq 1 \text{ for a.e. } \xi \in \mathbb{R}^N \tag{III.19}
\]

\[
\sum_{\omega \in \Omega} \mathcal{D}_\Psi(\xi + (A^t)^{-1}\omega) = |\Omega^\prime| \cdot L + \sum_{\omega' \in \Omega^\prime} \mathcal{D}_\Psi(A^t\xi + \omega') \text{ for a.e. } \xi \in \mathbb{R}^N \tag{III.20}
\]

\[
\sum_{k \in \mathbb{Z}^N} \mathcal{I}_\Delta(\xi + k) \geq \mathcal{D}_\Psi(\xi) \text{ for a.e. } \xi \in \mathbb{R}^N \tag{III.21}
\]
where $\Omega, \Omega'$ are transversals of $\Gamma/A^\perp Z^N$ and $\Gamma/Z^N$, respectively, with

$$\Gamma = Z^N + A^\perp Z^N,$$

and $\Delta = \{ \xi \in \mathbb{R}^N : \mathcal{D}_\Phi((A^\perp)^{-j} \xi) \geq 1 \text{ for all } j \in \mathbb{N} \cup \{0\} \}$. 

Proof. Condition (III.18) is verified by the argument

$$
\int_{T^N} \mathcal{D}_\Phi(\xi) d\xi = \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} \int_{\mathbb{R}^N/Z^N} |\psi^\ell((A^\perp)^j(\xi + k))|^2 d\xi
$$

$$
= \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} \int_{\mathbb{R}^N} |\psi^\ell((A^\perp)^j\xi)|^2 d\xi
$$

$$
= \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} |\text{det}(A)|^{-j} \int_{\mathbb{R}^N} |\psi^\ell(\xi)|^2 d\xi
$$

$$
= \sum_{j=1}^{\infty} |\text{det}(A)|^{-j} \cdot \sum_{\ell=1}^{L} \|\psi^\ell\|^2
$$

$$
= \frac{1}{|\text{det}(A)| - 1} \cdot L.
$$

Condition (III.19) follows from the fact that for each $n \in \mathbb{N}$ and $\xi \in \mathbb{R}^N$ we have
\[ \mathcal{D}_\psi((A^T)^{-n}\xi) = \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^N} |\hat{\psi}_\ell((A^T)^{j-n}\xi + (A^T)^{j}k)|^2 \]

\[ \geq \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} |\hat{\psi}_\ell((A^T)^{j-n}\xi)|^2 = \sum_{\ell=1}^{L} \sum_{j=1-n}^{\infty} |\hat{\psi}_\ell((A^T)^{j}\xi)|^2, \]

and so

\[ \liminf_{n \to \infty} \mathcal{D}_\psi((A^T)^{-n}\xi) \geq \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} |\hat{\psi}_\ell((A^T)^{j}\xi)|^2 = 1 \]

for almost every \( \xi \in \mathbb{R}^N \).

The consistency condition, (III.20), is given by

\[ \sum_{\omega \in \Omega} \mathcal{D}_\psi(\xi + (A^T)^{-1}\omega) = \sum_{\omega \in \Omega} \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^N} |\hat{\psi}_\ell((A^T)^{j}(\xi + (A^T)^{-1}\omega + k))|^2 \]

\[ = \sum_{\omega \in \Omega} \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^N} |\hat{\psi}_\ell((A^T)^{j}(A^T\xi + \omega + A^T)k))|^2 \]

\[ = \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \sum_{\gamma \in \Gamma} |\hat{\psi}_\ell((A^T)^{j}(A^T(\xi + m) + n))|^2 \]
\[
= \sum_{\omega' \in \Omega'} \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^N} |\tilde{\psi}^\ell((A^T)^j(A^T \xi + \omega' + k))|^2
\]

\[
= |\Omega'| \cdot L + \sum_{\omega' \in \Omega'} \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^N} |\tilde{\psi}^\ell((A^T)^j(A^T \xi + \omega' + k))|^2
\]

\[
= |\Omega'| \cdot L + \sum_{\omega' \in \Omega'} \mathcal{D}_\psi(A^T \xi + \omega'),
\]

where the penultimate equality holds for almost every \( \xi \in \mathbb{R}^N \) and is obtained by separating \( \sum_{\omega' \in \Omega'} \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^N} |\tilde{\psi}^\ell((A^T)^j(A^T \xi + \omega' + k))|^2 \) into

\[
\sum_{\omega' \in \Omega'} \sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}^N} |\tilde{\psi}^\ell(A^T \xi + \omega' + k)|^2 + \sum_{\omega' \in \Omega'} \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^N} |\tilde{\psi}^\ell((A^T)^j(A^T \xi + \omega' + k))|^2
\]

and invoking the fact that for each \( \omega' \in \Omega' \) and \( \ell = 1, \ldots, L \) we have

\[
\sum_{k \in \mathbb{Z}^N} |\tilde{\psi}^\ell(A^T \xi + \omega' + k)|^2 = 1
\]

for almost every \( \xi \in \mathbb{R}^N \).

Finally, to verify (III.21) define \( \gamma : \mathbb{R}^N \rightarrow [0, \infty) \) by

\[
\gamma(\xi) = \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} |\tilde{\psi}^\ell((A^T)^j \xi)|^2, \quad \text{so that} \quad \mathcal{D}_\psi(\xi) = \sum_{k \in \mathbb{Z}^N} \gamma(\xi + k). \quad \text{Notice that for each} \ n \ \text{in} \ \mathbb{Z} \ \text{we have}
\]

\[
\gamma((A^T)^n \xi) = \sum_{\ell=1}^{L} \sum_{j=n+1}^{\infty} |\tilde{\psi}^\ell((A^T)^j \xi)|^2. \quad (III.22)
\]
We claim that $\gamma(\xi) \leq 1_\Delta(\xi)$ for almost every $\xi \in \mathbb{R}^N$. Indeed,

$$\gamma(\xi) \leq \sum_{\ell=1}^L \sum_{j \in \mathbb{Z}} \left| \psi^j((A^\ell)^j \xi) \right|^2 = 1$$

almost everywhere, and so $\gamma(\xi) \leq 1_\Delta(\xi)$ for almost every $\xi \in \Delta$. Suppose, then, that $\xi \notin \Delta$. Then there exists some $j \geq 0$ such that $\mathcal{D}_\psi((A^j)^{-j} \xi) < 1$. Since $\mathcal{D}_\psi$ maps $\mathbb{R}^N$ into $\mathbb{N} \cup \{0\}$, it follows that $\mathcal{D}_\psi((A^j)^{-j} \xi)$ must equal 0. This implies, since $\gamma$ maps $\mathbb{R}^N$ into $[0, \infty)$ and

$$\sum_{k \in \mathbb{Z}^N} \gamma((A^j)^{-j} \xi + k) = \mathcal{D}_\psi((A^j)^{-j} \xi) = 0,$$

that $\sum_{k \in \mathbb{Z}^N} \gamma((A^j)^{-j} \xi + k) = 0$ for all $k \in \mathbb{Z}^N$. In particular, $\gamma((A^j)^{-j} \xi) = 0$. Hence, by (III.22) we have

$$0 \leq \gamma(\xi) \leq \gamma((A^j)^{-j} \xi) = 0$$

and, thus, $\gamma(\xi) = 0 = 1_\Delta(\xi)$.

Notice that this also proves the necessity of the conditions in Theorem III.2.3. Indeed, the only condition that appears different from Theorem III.2.3 is the consistency condition, but when $A \in M_N(\mathbb{Z})$ then $A^* \mathbb{Z}^N \subset \mathbb{Z}^N$ and, hence, $\Gamma = \mathbb{Z}^N$. Thus, $\Gamma / A^* \mathbb{Z}^N = \mathbb{Z}^N / A^* \mathbb{Z}^N$ and $\Gamma / \mathbb{Z}^N = \mathbb{Z}^N / \mathbb{Z}^N = \{0\}$. Therefore, when $A \in M_N(\mathbb{Z})$ (III.20) is identical to (III.5).

As for the sufficiency of the conditions in Theorem III.3.8, it is not presently known whether these four conditions are sufficient to characterize all dimension functions of rationally dilated wavelets. In particular, one can ask whether the algorithm for verifying sufficiency in the case of integer dilations holds when the dilation is allowed to be non-integer. In the next section we will present an example of a dimension function of a rationally dilated wavelet for which Algorithm III.2.5 fails to provide a wavelet with the given dimension function.
III.4 Applications and Examples

Example III.4.1.

Consider the wavelet set $W$ as given in Theorem II.3.4(i) with the parameters $a = \frac{11}{9}, p = 3, n = 5, m = 1$. That is,

$$W = \left[ \frac{-3311}{808}, \frac{-2709}{808} \right] \cup \left[ \frac{6561}{8080}, \frac{729}{808} \right] \cup \left[ \frac{1331}{808}, \frac{14641}{8080} \right].$$

Define the wavelet $\psi \in L^2(\mathbb{R})$ by $\hat{\psi} = 1_W$. We wish to calculate the dimension function of $\psi$. We have already shown in (III.10) that $D_\psi(\xi) = \sum_{k \in \mathbb{Z}} 1_S(\xi + k)$ where

$$S = \bigcup_{j=1}^{\infty} a^{-j}W = \left[ \frac{-2709}{808}, \frac{6561}{8080} \right] \cup \left[ \frac{729}{808}, \frac{809}{8080} \right] \cup \left[ \frac{891}{808}, \frac{981}{8080} \right] \cup \left[ \frac{1089}{808}, \frac{11979}{8080} \right].$$

Therefore, $D_\psi$ is given by

$$D_\psi(\xi) = \begin{cases} 4 & \text{if } \xi + k \in X \text{ for some } k \in \mathbb{Z} \\ 5 & \text{otherwise} \end{cases},$$

where $X = \left[ \frac{-1}{2}, \frac{-285}{808} \right] \cup \left[ \frac{-1519}{8080}, \frac{-79}{808} \right] \cup \left[ \frac{-61}{8080}, \frac{83}{808} \right] \cup \left[ \frac{1721}{8080}, \frac{281}{808} \right] \cup \left[ \frac{3899}{8080}, \frac{1}{2} \right]$. One period of $D_\psi$ is shown in Figure III.1.

We claim that Algorithm III.2.5 fails for this example. (While reading the following discussion, it may be helpful to follow along with Figure III.2 on page 61.) For each $i \in \{2,3,4\}$ let $N_i = \left\lfloor \log_\frac{1}{8} \left( \frac{i}{i-1} \right) \right\rfloor$. Notice that $N_2 = 4$, $N_3 = 3$, and $N_4 = 2$.

We begin the algorithm by defining $E_1 = [-\varepsilon, 1 - \varepsilon)$ for some $0 < \varepsilon < \frac{61}{8080} \cdot \left( \frac{11}{9} \right)^{-7}$. Clearly, $E_1$ satisfies conditions 1(i) 1(iv) of Algorithm III.2.5. Since $|E_1| = 1$, it follows that $E_1^p = \mathbb{R} and, hence, E_n = \emptyset$ for all $n \geq 2$. Thus, $P_1 = S_1 = [-\varepsilon, 1 - \varepsilon)$. Continuing with the algorithm, we have
Figure III.1: Dimension function $\mathcal{D}_\psi$ for Example III.4.1

$$F_{2,n} = \begin{cases} 
- \left(\frac{11}{9}\right)^n \varepsilon, -\left(\frac{11}{9}\right)^{n-1} \varepsilon \right) \cup \left[ \left(\frac{11}{9}\right)^{n-1} (1 - \varepsilon), \left(\frac{11}{9}\right)^n (1 - \varepsilon) \right] & \text{if } n < 4 \\
\left[ \left(\frac{11}{9}\right)^{3} (1 - \varepsilon), 2 - \left(\frac{11}{9}\right)^{3} \varepsilon \right] & \text{if } n = 4 \\
\emptyset & \text{if } n > 4 
\end{cases}$$

This gives $S_2 = \left[ - \left(\frac{11}{9}\right)^{3} \varepsilon, -\varepsilon \right] \cup \left[ 1 - \varepsilon, 2 - \left(\frac{11}{9}\right)^{3} \varepsilon \right]$ and, hence, $P_2 = \left[ - \left(\frac{11}{9}\right)^{3} \varepsilon, 2 - \left(\frac{11}{9}\right)^{3} \varepsilon \right]$. 

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Figure III.2: Applying Algorithm III.2.5 to Example III.4.1.
The next two iterations yield

\[
F_{3,n} = \begin{cases} 
  \left[ -\left(\frac{11}{9}\right)^{n+3} \epsilon, -\left(\frac{11}{9}\right)^{n+2} \epsilon \right] 
  & \text{if } n < 3 \\
  \left[ 2\left(\frac{11}{9}\right)^{n-1} - \left(\frac{11}{9}\right)^{n+2} \epsilon, 2\left(\frac{11}{9}\right)^n - \left(\frac{11}{9}\right)^{n+3} \epsilon \right] 
  & \text{if } n = 3 \\
  \emptyset 
  & \text{if } n > 3
\end{cases}
\]

\[
S_3 = \left[ -\left(\frac{11}{9}\right)^5 \epsilon, -\left(\frac{11}{9}\right)^3 \epsilon \right] \cup \left[ 2 - \left(\frac{11}{9}\right)^3 \epsilon, 3 - \left(\frac{11}{9}\right)^5 \epsilon \right]
\]

\[
P_3 = \left[ -\left(\frac{11}{9}\right)^5 \epsilon, 3 - \left(\frac{11}{9}\right)^5 \epsilon \right],
\]

and

\[
F_{4,n} = \begin{cases} 
  \left[ -\left(\frac{11}{9}\right)^{n+5} \epsilon, -\left(\frac{11}{9}\right)^{n+4} \epsilon \right] 
  & \text{if } n = 1 \\
  \left[ 3\left(\frac{11}{9}\right)^{n-1} - \left(\frac{11}{9}\right)^{n+4} \epsilon, 3\left(\frac{11}{9}\right)^n - \left(\frac{11}{9}\right)^{n+5} \epsilon \right] 
  & \text{if } n = 2 \\
  \left[ 3\left(\frac{11}{9}\right) - \left(\frac{11}{9}\right)^6 \epsilon, 4 - \left(\frac{11}{9}\right)^6 \epsilon \right] 
  & \text{if } n = 2 \\
  \emptyset 
  & \text{if } n > 2
\end{cases}
\]
\[ S_4 = \left[ -\left(\frac{11}{9}\right)^6 \varepsilon, -\left(\frac{11}{9}\right)^5 \varepsilon \right] \cup \left[ 3 - \left(\frac{11}{9}\right)^5 \varepsilon, 4 - \left(\frac{11}{9}\right)^6 \varepsilon \right) \]

\[ P_4 = \left[ -\left(\frac{11}{9}\right)^6 \varepsilon, 4 - \left(\frac{11}{9}\right)^6 \varepsilon \right). \]

Now, one must use caution when approaching the fifth iteration due to the fact that \( A_5^P \neq \mathbb{R} \), unlike \( A_2^P, \ldots, A_4^P \). We know that

\[ \widetilde{F}_{5,1} = \left( A_5^P \cap \left[ -\left(\frac{11}{9}\right)^7 \varepsilon, -\left(\frac{11}{9}\right)^6 \varepsilon \right] \right) \cup \left( A_5^P \cap \left[ 4 - \left(\frac{11}{9}\right)^6 \varepsilon, 4 - \left(\frac{11}{9}\right)^7 \varepsilon \right] \right). \]

Since the condition that \( \varepsilon \) be less than \( \frac{61}{8080} \cdot \left(\frac{11}{9}\right)^{-7} \) implies that \( -\frac{61}{8080} < -\left(\frac{11}{9}\right)^7 \varepsilon \), the above reduces to

\[ \widetilde{F}_{5,1} = A_5^P \cap \left[ 4 - \left(\frac{11}{9}\right)^6 \varepsilon, 4 - \left(\frac{11}{9}\right)^7 \varepsilon \right]. \]

Furthermore, we claim that

\[ 5 - \frac{1519}{8080} < 4\left(\frac{11}{9}\right)^7 \varepsilon < 5 - \frac{79}{808}. \]

Indeed, we have \( \varepsilon < \frac{61}{8080} \cdot \left(\frac{11}{9}\right)^{-7} < \frac{5591}{72720} \cdot \left(\frac{11}{9}\right)^{-7} \). This implies that \( \left(\frac{11}{9}\right)^7 \varepsilon < \frac{5591}{72720} = 4\left(\frac{11}{9}\right) - 5 + \frac{1519}{8080} \) and, hence, \( 5 + \frac{1519}{8080} < 4\left(\frac{11}{9}\right) - \left(\frac{11}{9}\right)^7 \varepsilon \). Also, since
\[ 4 \left( \frac{11}{9} \right) < \frac{3061}{808} = 5 - \frac{79}{808}, \text{ then } 4 \left( \frac{11}{9} \right) - \left( \frac{11}{9} \right)^7 < 5 - \frac{79}{808}. \] Therefore, we are left with

\[ F_{5,1} = \left[ 4 + \frac{83}{808}, 4 + \frac{1721}{8080} \right] \cup \left[ 4 + \frac{281}{808}, 4 + \frac{3899}{8080} \right] \cup \left[ 5 - \frac{285}{808}, 5 - \frac{1519}{8080} \right]. \]

Finally, we claim that \( F_{5,2} = \emptyset \) and, hence, \( F_{5,n} = \emptyset \) for all \( n \geq 2 \). This is true because \( 5 - \frac{61}{808} < \frac{11}{9} \left( 4 + \frac{83}{808} \right) \) and \( \frac{11}{9} \left( 5 + \frac{1519}{8080} \right) < 6 - \frac{79}{808} \). Therefore, the algorithm stops, giving the output

\[ S = [\delta, 4 - \delta] \cup \left[ 4 + \frac{83}{808}, 4 + \frac{1721}{8080} \right] \cup \left[ 4 + \frac{281}{808}, 4 + \frac{3899}{8080} \right] \cup \left[ 5 - \frac{285}{808}, 5 - \frac{1519}{8080} \right]. \]

where \( \delta = -\left( \frac{11}{9} \right)^{N_2 + N_3 + N_4 - 3} \varepsilon \). Notice that if \( S \) is, indeed, a generalized scaling set then any associated wavelet must satisfy \( |\tilde{\psi}| = 1_W \) where \( W = \frac{11}{9} S \setminus S \). Since the dimension function of \( \tilde{\psi} \) is then given by \( D_\psi(\xi) = \sum_{k \in \mathbb{Z}} 1_S(\xi + k) \) for each \( \xi \in \mathbb{R} \), we have

\[ D_\psi(\xi) = \begin{cases} 4 & \text{if } \xi + k \in \bar{X} \text{ for some } k \in \mathbb{Z} \\ 5 & \text{otherwise} \end{cases}, \]

where \( \bar{X} = \left[ \frac{-1}{2}, \frac{-285}{808} \right] \cup \left[ \frac{-1519}{8080}, \frac{83}{808} \right] \cup \left[ \frac{1721}{8080}, \frac{281}{808} \right] \cup \left[ \frac{3899}{8080}, \frac{1}{2} \right]. \) Thus, \( D_\psi \neq D_\psi \) and the algorithm fails. Compare Figures III.1 and III.3.

**Example III.4.2.**

A seminal concept in wavelet theory is that of a *multi-resolution analysis* or MRA. A multi-resolution analysis is, by definition, a sequence \( \{ V_j \}_{j \in \mathbb{Z}} \) of closed
subspaces of $L^2(\mathbb{R}^N)$ that satisfy the following properties:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$

(ii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$

(iii) $\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}^N)$

(iv) $f \in V_0$ if and only if $D_j f \in V_j$

(v) $V_0$ is shift invariant

(vi) there exists $\varphi \in V_0$ such that $\{T_k \varphi : k \in \mathbb{Z}^N\}$ is an orthonormal basis of $V_0$

If, in the above definition, (v) is replaced with
there exists \( \{\varphi^1, \ldots, \varphi^K\} \subset V_0 \) such that \( \{T_k\varphi^\ell : k \in \mathbb{Z}^N; \ell = 1, \ldots, L\} \)

is an orthonormal basis of \( V_0 \)

then the sequence \( \{V_j\}_{j \in \mathbb{Z}} \) is called a multi-resolution analysis of multiplicity \( K \). If no multiplicity is mentioned, then it is understood to be 1.

If \( \Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^N) \) is a wavelet such that the spaces

\[
V_j = \overline{\text{span}} \{\psi^\ell_{j',k} : j' < j; k \in \mathbb{Z}^N; \ell = 1, \ldots, L\}
\]

form a multi-resolution analysis, then we say that \( \Psi \) is an MRA wavelet (or, alternatively, belongs to an MRA or is associated with an MRA).

It has long been known that a wavelet \( \Psi \) is an MRA wavelet if and only if \( \mathcal{D}_\Psi = 1 \) almost everywhere. Furthermore, it is known that the wavelet \( \Psi \) belongs to an MRA of multiplicity \( K \) for some \( K \in \mathbb{N} \) if and only if \( \mathcal{D}_\Psi = K \) almost everywhere (see [9], Theorem 1.5). We will now show the following:

**Theorem III.4.3.** Every two interval MSF wavelet belongs to an MRA of multiplicity \( n \) for some \( n \in \mathbb{N} \).

**Remark III.4.4.** Note that we are not using Definition III.3.1 since we are dealing with real dilations. However, Theorem 3.2 of [12] shows that the dimension function for any MSF wavelet in \( L^2(\mathbb{R}) \) agrees with Definition III.3.1.

**Proof.** Let \( \psi \in L^2(\mathbb{R}) \) be a two interval MSF wavelet. That is, \( \psi \) is a wavelet with \( |\hat{\psi}| = 1_W \) for some \( W \subset \mathbb{R} \) consisting of two intervals. Theorem II.3.1 shows that
the dilation \( \alpha \) associated with \( \psi \) is of the form \( \alpha = \frac{n+1}{n} \) for some \( n \in \mathbb{N} \), and that
\[
W = [ax, x] \cup \left[ x + \frac{1}{a-1}, ax + \frac{a}{a-1} \right] \text{ for some } x \in \left( \frac{-1}{a-1}, 0 \right).
\]
Furthermore we have already seen that when \( \psi \) is an MSP wavelet then \( D_\psi(\xi) = \sum_{k \in \mathbb{Z}} 1_S(\xi + k) \) where \( S + \bigcup_{j=1}^{\infty} \alpha^{-j} W \). Therefore, we have
\[
D_\psi(\xi) = \sum_{k \in \mathbb{Z}} 1_{[x,x+n]}(\xi + k) = n \text{ for a.e. } \xi \in \mathbb{R}
\]
Thus, \( \psi \) belongs to an MRA of multiplicity \( n \).

Example III.4.5.

In [11], Theorem 5.11, Bownik, Rzeszotnik and Spegle prove that the dimension function of an integer dilated wavelet must take on all values from 1 to its maximum value unless it is a constant function with value greater than 1 (that is, the wavelet belongs to an MRA of multiplicity greater than 1). In this example, we show that this is not true (at least in the case \( N = 1 \)) for rationally dilated wavelets. This was actually already accomplished in Example III.4.1 with the construction of a dimension function whose range is \( \{4,5\} \). However, our goal here is to show that given any a priori lower bound \( K \in \mathbb{N} \), one can easily construct a wavelet whose dimension function is not constant but respects this lower bound.

Once again, we employ Theorem II.3.4(i). Let \( \psi \) be given by \( \hat{\psi} = 1_W \) where \( W \) is as in Theorem II.3.4(i). Recall that the generalized scaling set associated with \( W \) is \( S = \bigcup_{j=1}^{\infty} \alpha^{-j} W \). In this case we have
\[ S = \left[ \frac{m - n(a^p - 1) - 1}{a^p + 1 - 1}, \frac{m}{a^p + 1 - 1} \right] \cup \bigcup_{r=1}^{p} a^{-r} \left[ \frac{a^p (a(m - 1) + n(a - 1))}{a^{r+1} + 1}, \frac{a^{p+1} m}{a^{r+1} + 1} \right] \]

\[ = \left[ \frac{m - n(a^p - 1) - 1}{a^p + 1 - 1}, \frac{m}{a^p + 1 - 1} \right] \cup \bigcup_{r=0}^{p-1} a^{-r} \left[ \frac{a^r (a(m - 1) + n(a - 1))}{a^{r+1} + 1}, \frac{a^{r+1} m}{a^{r+1} + 1} \right] \]

Notice that the length of the first interval is \( \frac{n(a^p - 1) + 1}{a^{p+1} - 1} \). Since
\[ \mathcal{D}_{\psi}(\xi) = \sum_{k \in \mathbb{Z}} \mathbb{1}_S(\xi + k) \text{ for all } \xi \in \mathbb{R}, \quad \mathcal{D}_{\psi}(\xi) \geq \left[ \frac{n(a^p - 1) + 1}{a^{p+1} - 1} \right] \text{ for each } \xi \in \mathbb{R}. \]

Suppose we are given \( K \in \mathbb{N} \). Choose \( a \in \mathbb{Q} \) such that \( 1 < a < \sqrt{\frac{K+2}{K}} \) and \( a \neq \frac{k+1}{k} \) for all \( k \in \mathbb{N} \). We have shown in Section 2.3.3 that \( \left( 1, \left[ \frac{a}{a-1} \right] \right) \in \mathcal{F}_1(a, 1) \).

Thus, we may choose \( p = 1, m = 1, n = \left[ \frac{a}{a-1} \right] \). This yields
\[ W = \left[ \frac{\frac{n}{a+1}, \frac{n}{a+1}}{a^2 - 1}, \frac{n}{a+1} \right] \cup \left[ \frac{\frac{n}{a+1}, \frac{a^2}{a^2 - 1}}{a^2 - 1} \right] \text{ and } S = \left[ \frac{\frac{n}{a+1}, \frac{1}{a^2 - 1}}{a^2 - 1}, \frac{n}{a+1}, \frac{a}{a-1} \right]. \]

Since \( n = \left[ \frac{a}{a-1} \right] \), it follows that \( n > -\frac{a}{a-1} - 1 = \frac{1}{a-1} \). This, along with the fact that
\[ a < \sqrt{\frac{K+2}{K}}, \]
implies that \( K(a^2 - 1) - n(a - 1) < K(a^2 - 1) - 1 < 1 \). This yields
\[ K < \frac{n(a-1) + 1}{a^2 - 1} \]
and, hence, for each \( \xi \in \mathbb{R} \) we have \( \mathcal{D}_{\psi}(\xi) \geq \left[ \frac{n(a-1) + 1}{a^2 - 1} \right] \geq K. \)

Therefore, all that remains to be shown is that \( \mathcal{D}_{\psi} \neq K \) (that is, \( \psi \) does not belong to an MRA of multiplicity \( K \)). This can be done by showing that \( |S| \notin \mathbb{Z} \) since \( \mathcal{D}_{\psi}(\xi) = \sum_{k \in \mathbb{Z}} \mathbb{1}_S(\xi + k) \). Indeed, notice that \( |S| = \frac{1}{a-1} \). Thus, since \( a \neq \frac{k+1}{k} \) for all \( k \in \mathbb{N} \), it follows that \( |S| \notin \mathbb{Z} \).

As an example, suppose we wish to produce a non-MRA wavelet whose dimension function does not assume any values less than the 100th prime number.

That is, we wish \( \mathcal{D}_{\psi} \) to be non-constant with \( \mathcal{D}_{\psi}(\xi) \geq 541 \) for all \( \xi \in \mathbb{R} \). Let us use the dilation \( a = \frac{2004001}{2002000} < \sqrt{\frac{543}{541}} \). As stated earlier, we use \( p = 1, m = 1, \)
\[ n = \left[ \frac{a^2}{a-1} \right] = 1001. \] These parameters give
\[ W = \left[ \begin{array}{c} -2006005001 \\ 4066001 \\ -2004002000 \\ 4066001 \end{array} \right] \cup \left[ \begin{array}{c} 4068004000000 \\ 8016008001 \end{array} \right] \cup \left[ \begin{array}{c} 2004002000 \\ 4066001 \end{array} \right] \cup \left[ \begin{array}{c} 2006005001 \\ 40160020008001 \end{array} \right] \]

and

\[ S = \left[ \begin{array}{c} -2004002000 \\ 4066001 \end{array} \right] \cup \left[ \begin{array}{c} 2004002000 \\ 4066001 \end{array} \right] \cup \left[ \begin{array}{c} 2012001002000 \\ 8016008001 \end{array} \right]. \]

Figure III.4: Dimension function for Example III.4.5

If \( \psi \) is given by \( \hat{\psi} = I_W \) with

\[ X = \left[ \begin{array}{c} -1001500 \\ 4066001 \end{array} \right] \cup \left[ \begin{array}{c} -500 \\ 8016008001 \end{array} \right] \cup \left[ \begin{array}{c} 1001500 \\ 4066001 \end{array} \right] \cup \left[ \begin{array}{c} 4006001500 \\ 8016008001 \end{array} \right] \]

then the dimension function of \( \psi \) is given by

\[
D_\psi(\xi) = \begin{cases} 
1001 & \text{if } \xi + k \in X \text{ for some } k \in \mathbb{Z} \\
1000 & \text{otherwise}
\end{cases}
\]

Notice that this dimension function far surpasses the lower bound condition that we asked of it. This demonstrates how a slight decrease in \( a \) greatly affects the
values of $D_\phi$. Indeed, while the choice of dilation used in this example was somewhat conservative, it is worth noting that $\sqrt{\frac{543}{541}} - \frac{2004001}{2002000} < 0.00085$. 

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INDEX

affine system, 2
Auscher's geometric lemma, 44
celling see least integer, 1
consistency condition, 50
core space, 3
dilation, 2
  integer . . . , 2
  rational . . . , 2
dimension function, 39
  . . . of a rationally dilated wavelet, 50
  . . . of an integer dilated wavelet, 42
floor see greatest integer, 1
Fourier transform, 1
frequency domain, 7

generalized scaling set, 31

greatest integer, 1
lattice, 2
least integer, 1

multi-resolution analysis (MRA), 72

oversampled system, 57

Parseval frame, 40
Plancherel's Theorem, 1
quasi-affine system, 57
range function, 39
shift invariant . . . , 38
  . . . space, 38
  . . . space generated by, 38
  . . . system generated by, 38
spectral function, 40
torus, 1
translation projection, 11
transversal, 2
wavelet, 3
  . . . set, 8
  minimally supported frequency
    (MSF) . . . , 4
    MRA . . . , 74
REFERENCES


