1	2	3	4	5	6	7	8	9	10	
11	12	13	14	15	16	17	18	19	20	Total

QUALIFYING EXAM, Winter 2014

Algebraic Topology and Differential Geometry

NAME ____

(PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ____________________ **SIGNATURE** ____________________

Please do any 10 problems out of the following 20.

1. Algebraic Topology

Problem 1.1. Define Eilenberg-McLane space $K(\pi, n)$. Prove that $H_{n+1}(K(\pi,n);\mathbf{Z}) = 0$ for $n \geq 2$ and an arbitrary abelian group π.

Problem 1.2. Let M be a closed, orientable manifold of dimension $4k+2$. Show that the Euler characteristic of M is even.

Problem 1.3. Define the Hopf invariant. Assume the Hopf invariant is a homomorphism. Prove that $h([\iota_{2n}, \iota_{2n}])$ is non-zero, and use this to prove that $\pi_{4n-1}(S^{2n})$ contains $\mathbf{Z}$.

Problem 1.4. Let $f : S^n \times S^n \to S^{2n}$ be the quotient map collapsing $S^n \vee S^n$ to a point. Show that f induces trivial homomorphism on all homotopy groups but f is not nullhomotopic.

Problem 1.5. State the Lefschetz Fixed Point Theorem. Let

$$f : \mathbf{CP}^{4k} \times \mathbf{RP}^{2n} \to \mathbf{CP}^{4k} \times \mathbf{RP}^{2n}$$

be a map. Prove that f always has a fixed point.

Problem 1.6. Prove the spaces $\mathbf{RP}^n \times S^k$ and $S^n \times \mathbf{RP}^k$ are homotopy equivalent if and only if $k = n$.

Problem 1.7. State Jordan-Brouwer Theorem. Let $K \subset S^n$, be homeomorphic to the cube I^m, where $0 \leq m \leq n-1$. Prove that

$$\widetilde{H}_q(S^n \setminus K) = 0 \text{ for all } q \geq 0.$$

Problem 1.8. Prove that any map $f : \mathbf{RP}^2 \times \mathbf{RP}^2 \times \mathbf{RP}^2 \to T^6$ is homotopic to a constant map.

Problem 1.9. Compute the homotopy groups $\pi_q(\mathbf{RP}^2 \vee S^2)$ for $q = 1, 2$.

Problem 1.10. Let $p : E \to B$ be a Serre fiber bundle, where B is a path connected space. Prove that for any two points $x_0, x_1 \in B$ the fibers $F_0 = p^{-1}(x_0)$ and $F_1 = p^{-1}(x_1)$ are weak homotopy equivalent.

2. Differential Geometry

Problem 2.1. Let M the smooth submanifold of $\mathbb{C}^{n+1}$ defined by

$$\{(z_0, \cdots, z_n) : \sum z_i^2 = 1.\}$$

Prove that M is diffeomorphic to the tangent bundle of the unit sphere TS^n.

Problem 2.2. Recall that a manifold (M^{2n}, ω) is called a *symplectic manifold* if ω is a closed two-form ($d\omega = 0$) and ω is non-degenerate, namely ω^n is a nowhere vanishing $2n$-form.

(1) Can ω be an exact form if M is compact? What if the compactness condition is dropped? Justify your answer.
(2) When does the unit sphere $S^{2n} \subset \mathbb{R}^{2n+1}$ admit a symplectic structure? Justify your answer.

Problem 2.3. Let $M = f^{-1}(c)$ be the manifold defined by the level set of $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with $df \neq 0$ (at the level set). Constructing a nowhere vanishing n-form on M to show that M is orientable.

Problem 2.4. A smooth manifold of dimension n has trivial tangent bundle if there exists a set of n vector fields $\{X_1, \cdots, X_n\}$ forms a basis for the tangent space at any point; otherwise it has nontrivial tangent bundle. Show that the unit sphere $S^{2m} \subset \mathbb{R}^{2m+1}$ of even dimension has nontrivial tangent bundle.

Problem 2.5. Let (M, g) be a complete simply connected Riemannian manifold of non-positive sectional curvature. Let T be an isometry of (M, g) satisfying $T^2 = \mathrm{Id}$. Show T has a fixed point.

Problem 2.6. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = -1, z > 0\}$ be given the metric induced from the Minkowski metric on $\mathbb{R}^3$ given by $dx^2 + dy^2 - dz^2$. Show that S is a complete, simply connected Riemannian manifold of constant sectional curvature -1.

Problem 2.7. Let $N = \mathbb{R}^2 - \{0\}$ with the Lorentz metric $d = dx \circ dy/(x^2 + y^2)$. The map $T : (x, y) \to 67(x, y)$ is an isometry of N and generates a fixed point free cyclic action; the quotient $N/\mathbb{Z} := M$ is a compact manifold isometric to the torus. Show that the compact quotient M is geodesically incomplete.

Problem 2.8. Prove or disprove the assertion: "*The Lie algebra of $SU(n)$ is the set of all complex $n \times n$ matrices A so that $Tr(A) = 0$ and $A + A^* = 0$.*"

Problem 2.9. Let G be the set of all matrices of the form

$$G = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} : 0 < a \in \mathbb{R} \text{ and } b \in \mathbb{R} \right\}.$$

Let $\mathfrak{g}$ be the Lie algebra of a connected Lie group G; identify $\mathfrak{g}$ with the left invariant vector fields on G. Let $\{e_i\}$ be a basis for $\mathfrak{g}$ and let $\{e^i\}$ be the dual basis for the space of left invariant 1-forms $\mathfrak{g}^*$. Let

$$H^p(\mathfrak{g}) := \frac{\ker(d : \Lambda^p(\mathfrak{g}^*) \to \Lambda^{p+1}(\mathfrak{g}^*))}{\mathrm{Range}(d : \Lambda^{p-1}(\mathfrak{g}^*) \to \Lambda^p(\mathfrak{g}^*))}.$$

You may use any of the results established in Math 638/9. But you must cite those results carefully if you use them.

(1) Determine $H^*_{\mathrm{DeR}}(G)$.
(2) Determine $H^*(\mathfrak{g})$.

Problem 2.10. Let (M, g) be a compact connected Riemannian manifold without boundary. Suppose that $\mathrm{Ric} \geq 0$ and that $\mathrm{Ric} > 0$ at some point P. Show that $H^{m-1}_{\mathrm{DeR}}(M) = 0$.