1	2	3	4	5	6	7	8	9	10	
11	12	13	14	15	16	17	18	19	20	Total

QUALIFYING EXAM, Winter 2012

Algebraic Topology and Differential Geometry

NAME ____

(PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ____________________ SIGNATURE ____________________

Please do any 10 problems out of the following 20.

1. Let $f : S^n \to S^n$ be a map, and $\deg(f)$ be the degree of f. Prove that

$$\text{Lef}(f) = 1 + (-1)^n \deg(f).$$

2. Define the Hopf invariant. Prove that the Hopf invariant is a homomorphism.

3. Consider the map

$$g : S^{2n-2} \times S^3 \xrightarrow{\text{proj}} (S^{2n-2} \times S^3)/(S^{2n-2} \vee S^3) = S^{2n+1} \xrightarrow{\text{Hopf}} \mathbf{CP}^n.$$

Prove that g induces trivial homomorphism in homology and homotopy groups, however g is not homotopic to a constant map.

4. Let $A \subset X$, and (X, A) be a Borsuk pair (for example, a CW-pair). Let $E = \mathcal{C}(X, Y)$, $B = \mathcal{C}(A, Y)$, and the map $p : E \to B$ be defined as $p(f : X \to Y) = (f|_A : A \to Y)$. Prove that the map $p : E \to B$ is a Serre fiber bundle.

5. Let $f : \mathbf{RP}^{2n} \to \mathbf{RP}^{2n}$ be a map. Prove that f always has a fixed point. Give an example that the above statement fails for a map $f : \mathbf{RP}^{2n+1} \to \mathbf{RP}^{2n+1}$.

6. Compute the homotopy group $\pi_3(S^2 \vee S^2)$.

7. Let $X = \mathbf{CP}^n$. Prove that $X|_3 = X|_{2n+1} = S^{2n+1}$.

8. Define regular covering. Describe all three-fold coverings of the figure 8. Identify which are regular and which are not.

9. Let X be path connected space. Prove that $\pi_1(X, x_0) = 0$ if and only if for every pair of points, $x, y \in X$, all paths from x to y are path homotopic.

10. Let π, π' be abelian groups, and n be a positive integer. Prove that there is a bijection

$$[K(\pi, n), K(\pi', n)] \leftrightarrow \text{Hom}(\pi, \pi').$$