Topology qualifying exam
Winter 2011
University of Oregon

1. Suppose that there is a fiber bundle $p\colon X \to S^7$ with fiber S^4. Prove that X is an orientable manifold and calculate its integral homology groups.

2. Let $n > 1$ and fix a nonzero vector $e \in \mathbb{C}^n$. Prove that there is no $\mathbb{C}$-bilinear map $\mu\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ with the properties that

   (i) $\mu(x,y) = 0 \iff (x = 0 \text{ or } y = 0)$;

   (ii) $\mu(e,x) = \mu(x,e) = x$ for all $x \in \mathbb{C}^n$.

3. Prove that the manifolds $\mathbb{C}P^3 \times S^8 \times S^{16}$ and $\mathbb{C}P^7 \times S^{16}$ are not homeomorphic.

4. Let M be a triangulable 3-manifold with $\pi_1(M) \cong \mathbb{Z}/5$. Let X be obtained from $\mathbb{R}P^4$ by attaching a single 10-cell along a map $S^9 \to \mathbb{R}P^4$. Prove that all maps from M to X are null homotopic.

5. Let $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$ and let $G = \pi_1(X, b)$ where b is the wedge point.

   (a) Draw a picture of the universal covering space $\tilde{X} \to X$.

   (b) Describe all covering spaces $p\colon E \to X$ where E is path-connected and $p_*(\pi_1(E)) \subseteq [G,G]$, where $[G,G] \subseteq G$ is the commutator subgroup. [Suggestion: Identify G and then compute $[G,G]$.]

6. Let S^1 be the unit complex numbers, and let $T = S^1 \times S^1$. Let $h\colon T \to T$ be the automorphism $(z,w) \mapsto (z, \bar{w})$. Finally, construct a space X by

$$X = (T \times I)/\sim$$

   where the equivalence relation is $(x,0) \sim (h(x),1)$. Calculate $\pi_i(X)$ and $H_i(X)$ for all $i \geq 1$.

7. Let M be a compact 2-manifold admitting a map $p\colon M \to S^2$ such that

   (i) There exist points $x, y, z \in S^2$ such that $p^{-1}(S^2 - \{x,y,z\}) \to S^2 - \{x,y,z\}$ is a 5-fold covering space, and

   (ii) Each of $p^{-1}(x)$, $p^{-1}(y)$, and $p^{-1}(z)$ is a singleton set.

   Determine what familiar 2-manifold M is homeomorphic to.

8. Let $T = \{a,b,c\}$, and let $T^n = T \times \cdots \times T$ (n factors). Define a chain complex by $C_n = \mathbb{Z}\langle T^{n+1}\rangle$ with $d\colon C_n \to C_{n-1}$ defined on basis elements by the formula

$$d([t_0, t_1, \ldots, t_n]) = \sum_{i=0}^{n} (-1)^i [t_0, \ldots, \widehat{t_i}, \ldots, t_n].$$

   Prove that C is chain homotopy equivalent to the chain complex $\mathbb{Z}[0]$ which has $\mathbb{Z}$ in dimension zero and 0 in all other dimensions.

9. (a) Give a Δ-complex model for $X = \mathbb{R}P^2 \# T$ (where T is the torus).

   (b) Let $p\colon X \to \mathbb{R}P^2$ be the map that squashes the torus part to a point, and let $\sigma = p^*(x)$ where $x \in H^1(\mathbb{R}P^2; \mathbb{Z}/2)$ is the nonzero element. Using your Δ-complex model from (a), describe σ explicitly as a cocycle.

   (c) Compute $(\sigma \cup \sigma) \cap [X]$, where $[X]$ is the fundamental class.

10. Let X be a simply-connected CW-complex such that $H_2(X) = 0$. Prove that every map $f\colon \mathbb{C}P^2 \to X$ is homotopic to a map sending $\mathbb{C}P^1$ to a point.