1. Let $QP^2$ be the quaternionic projective space, which has cohomology with an additive basis of $1, y_4, y_4^2$. Show that any map from $X = QP^2 \times CP^2$ to itself must have a fixed point.

2. Compute the homology groups of the manifold $M$ obtained from $S^1 \times S^2$ by identifying $(x \times y) \sim (-x \times -y)$, where by $-x$ and $-y$ we mean the antipodal points to $x$ and $y$ respectively. [Hint: this is feasible with cellular homology; start with the cell structures on spheres with two cells in each dimension.] Verify Poincaré duality for $H_\ast(M; \mathbb{Z}/2)$.

3. (a) Consider three copies of $S^2$, which we denote $S^2_{(i)}$, with three specified points on each, namely $a_{(i)}$, $b_{(i)}$ and $c_{(i)} \in S^2_{(i)}$. Compute the homology groups of $X$, which is the union of these $S^2$’s with identifications $a_{(1)} \sim a_{(2)} \sim a_{(3)}$ and $b_{(1)} \sim c_{(2)}$, $b_{(2)} \sim c_{(3)}$ and $b_{(3)} \sim c_{(1)}$.

(b) Compute the cohomology groups of $Y$ which is the complement in $\mathbb{R}^4$ of the planes $P_1 = \{(x,y,z,w)|x = y = 0\}$, $P_2 = \{(x,y,z,w)|z = w = 0\}$ and $P_3 = \{(x,y,z,w)|x + z = 1 = y + w\}$.

4. (a) Show, using only the definition of tensor product, that tensoring with $\mathbb{Q}$ preserves exact sequences of abelian groups.

(b) Use the fiber sequences $SU(n-1) \to SU(n) \to S^{2n-1}$ (obtained by letting $SU(n)$ act on $S^{2n-1}$ as the unit sphere in $\mathbb{C}^n$, which has stabilizer $SU(n-1)$) to compute $\pi_\ast(SU(n)) \otimes \mathbb{Q}$ for $n \geq 2$. You may assume that $\pi_\ast(S^{2n-1}) \otimes \mathbb{Q}$ is $\mathbb{Q}$ if $\ast = 2n - 1$ and zero otherwise.

5. Explicitly define Thom forms which generate $H^1(\mathbb{R}^3 - X)$ where $X$ is the union of the coordinate axes, proving that they span. You may assume the existence of real-valued functions with properties you specify.

6. (a) Let $M(\mathbb{Z}/n, 1)$ denote the Moore space obtained by attaching $D^2$ to $S^1$ using the map from the boundary of $D^2$ to $S^1$ sending $z$ to $z^n$. Explicitly construct a map $M(\mathbb{Z}/2^i, 1) \to M(\mathbb{Z}/2^{i+1}, 1)$ which on $H_1$ gives the standard inclusion of $\mathbb{Z}/2$ in $\mathbb{Z}/2^{i+1}$.

(b) Construct a space with $H_1 \cong \mathbb{Z}/2^\infty = \mathbb{Z}[1/2] / \mathbb{Z}$.

7. Let $\mathcal{D}$ be a partially ordered set, viewed as a category with either one or no morphism between any two objects, and let $\mathcal{C}$ be a category in which all colimits exist. Let $\text{Fun}(\mathcal{D}, \mathcal{C})$ be the category whose objects are functors from $\mathcal{D}$ to $\mathcal{C}$ and whose morphisms are natural transformations.
(a) Complete the following definition with a single equality. Let $F$ and $G$ be two functors from $\mathcal{D}$ to $\mathcal{C}$, and $\eta : F \to G$ a natural transformation. Let $i \in \mathcal{D}$ and let $f_i$ denote the canonical map from $F(i)$ to colim $F$, the colimit of $F$ (similarly, let $g_i : G(i) \to \text{colim } G$). Then the map $\eta_* : \text{colim } F \to \text{colim } G$ induced by $\eta$ is the unique map which for all $i$ satisfies:

(b) Explain why the map $\eta_*$ from part (a) exists, and then show that with this definition taking $F$ to the colimit of $F$ defines a functor from $\text{Fun}(\mathcal{D}, \mathcal{C})$ to $\mathcal{C}$.

8. Show that the unit tangent bundle of $S^2$ is homeomorphic to $\mathbb{R}P^3$. Use this fact (whether you can prove it or not) to show that there are no non-vanishing vector fields on $S^2$.

9. Is there a space whose integral cohomology groups are given by $H^0 \cong \mathbb{Z}$, $H^1 \cong 0$, $H^2 \cong \mathbb{Z}/2$, $H^3 \cong \mathbb{Z}/3$ and $H^4 \cong \mathbb{Z}/6$? Justify your answer. Repeat the question where $H^*(-; \mathbb{Z})$ is replaced by $H^*(-; \mathbb{Z}/6)$.
Name: 

1. Let $QP^2$ be the quaternionic projective space, which has cohomology with an additive basis of $1, y_4, y_4^2$. Show that any map from $X = QP^2 \times CP^2$ to itself must have a fixed point.

**Solution:** We use the Lefshetz theorem using $H^*(\mathbb{C}P^2; \mathbb{Z}/2)$. If we let $x_2$ be a generator of $H^2(\mathbb{C}P^2)$ then we have by the Künneth theorem that $x$ and $y$ together generate $H^*(X)$, since the cohomology of $QP^2$ and $CP^2$ are torsion-free. Also because they are torsion free, we can use their mod-2 reductions as generators for $H^*(X; \mathbb{Z}/2)$. We see that under some map $f : X \to X$, $f_*$ sends $x$ to $ax$ for some $a$. On the other hand $y$ maps to $by + cz^2$ for some $b$ or $c$ (in fact, one of these must be zero, but we will not need that). Then the trace in each dimension is calculated as follows:

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<tr>
<th>dim</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>tr</td>
<td>1</td>
<td>a</td>
<td>$a^2 + b$</td>
<td>$ab$</td>
<td>$b^2 + a^2b$</td>
<td>$ab^2$</td>
<td>$a^2b^2$</td>
</tr>
</tbody>
</table>

Because each of these dimensions is even, we add these traces to get the Lefshetz number. Working modulo two, we use the identity $x^2 = x$ repeatedly to see that the Lefshetz number is one, guaranteeing a fixed point.

2. Compute the homology groups of the manifold $M$ obtained from $S^1 \times S^2$ by identifying $(x \times y) \sim (-x \times -y)$, where by $-x$ and $-y$ we mean the antipodal points to $x$ and $y$ respectively. [Hint: this is feasible with cellular homology; start with the cell structures on spheres with two cells in each dimension.] Verify Poincaré duality for $H_*(M; \mathbb{Z}/2)$.

**Solution:** Put a cell structure on $S^1$ with two zero cells called $0_+ = (0, 1) \in \mathbb{R}^2$ and $0_- = (0, -1) \in \mathbb{R}^2$ and two one-cells $1_+ = \{(x, y)|x > 0\}$ and $1_- = \{(x, y)|x < 0\}$ [it would suffice to draw a picture]. Put a similar cell structure on $S^2$. Then $M$ has a cell structure with

| 3-cells | $1_+ \times 2_+ \sim 1_- \times 2_-, 1_+ \times 2_- \sim 1_- \times 2_+ $ |
| 2-cells | $1_+ \times 1_+, 1_+ \times 1_-, 0_+ \times 2_+, 0_+ \times 2_-$ |
| 1-cells | $1_+ \times 0_+ \sim 1_- \times 0_-, 1_+ \times 0_-, 0_+ \times 1_+, 0_+ \times 1_- $ |
| 0-cells | $0_+ \times 0_+, 0_+ \times 0_- $ |

If we choose the standard orientations on our original cell structures such that in those cellular chain complexes, for example, $d(2_+) = 1_+ + 1_- = d(2_-)$ while $d(1_+) = 0_+ - 0_- = -d(1_-)$, then the identifications such as $1_+ \times 1_- \sim 1_- \times 1_+$ are orientation
preserving. We thus may freely use the Leibniz rule to compute boundaries in the cellular complex of $M$, so for example

$$d(1_+ \times 2_+) = 0_+ \times 2_+ - 0_- \times 2_+ - 1_+ \times 1_+ - 1_- \times 1_-.$$ 

We omit the rest of these standard calculations from our solution. The most complicated boundary homomorphism is from $C^G_2(M)$ to $C^G_1(M)$, which when using bases as listed above is represented by the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}.$$ 

We know $H_0(M) \cong \mathbb{Z}$ because it is connected.

For $H_1$, we see cycles spanned by $1_+ \times 0_+ + 1_- \times 0_+$, $1_+ \times 0_+ - 0_+ \times 1_+$ and $0_+ \times 1_+ + 0_+ \times 1_-$. The first cycle is $d$ of $2_+ \times 0_+$ (and the additive inverse of $d$ of $2_- \times 0_+$) and the difference of the first and third is $d$ of $1_+ \times 1_+$. The image of $d$ is generated by these two cases. Thus, $H_1 \cong \mathbb{Z}$.

For $H_2$ we see cycles spanned by $a = 1_+ \times 1_+ + 1_+ \times 1_-$ and $b = 0_+ \times 2_+ - 0_+ \times 2_-$. But $d(1_+ \times 2_+) = b - a$ and $d(1_+ \times 2_-) = -b - a$. Thus $H_2 \cong \mathbb{Z}/2$.

Because $d$ is injective on $C^G_3(M)$, $H_3 = 0$.

With $\mathbb{Z}/2$ coefficients we get $H_3 \cong H_0 \cong \mathbb{Z}/2 \cong H_2 \cong H_1$, satisfying Poincaré duality.

3. (a) Consider three copies of $S^2$, which we denote $S^2(i)$, with three specified points on each, namely $a(i)$, $b(i)$ and $c(i) \in S^2(i)$. Compute the homology groups of $X$, which is the union of these $S^2$'s with identifications $a(1) \sim a(2) \sim a(3)$ and $b(1) \sim c(2)$, $b(2) \sim c(3)$ and $b(3) \sim c(1)$.

**Solution:** After the first identification we have $\vee_3 S^2$, which has reduced homology of rank three in degree two and zero otherwise. When the identification $b(1) \sim c(2)$ is made, we consider the corresponding sequence $S^0 \to \vee_3 S^2 \to \mathbb{Z} = \vee_3 S^2/(b(1) \sim c(2))$. The interesting part of the long exact sequence in reduced homology reads

$$\ldots \to \tilde{H}_1(\vee_3 S^2) = 0 \to \tilde{H}_1(Z) \to \tilde{H}_0(S^0) \cong \mathbb{Z} \to \tilde{H}_0(\vee_3 S^2) = 0.$$ 

So we get $H_0(Z) \cong H_1(Z) \cong \mathbb{Z}$ and $H_2(Z) \cong \mathbb{Z}^3$. The analysis goes through in exactly the same manner for the further two identifications, yielding $H_0(X) \cong \mathbb{Z}$ and $H_1(X) \cong H_2(X) \cong \mathbb{Z}^3$. 
(b) Compute the cohomology groups of $Y$ which is the complement in $\mathbb{R}^4$ of the planes $P_1 = \{(x, y, z, w) | x = y = 0\}$, $P_2 = \{(x, y, z, w) | z = w = 0\}$ and $P_3 = \{(x, y, z, w) | x + z = 1 = y + w\}$.

**Solution:** Consider $S^4$ as the one-point compactification of $\mathbb{R}^4$, so that the complement of the one-point compactification of the planes in $S^4$ is homeomorphic to the complement of the planes themselves in $\mathbb{R}^4$. But the one-point compactification of the planes is homeomorphic to $X$ from part (a): each plane with one point at infinity yields a copy of $S^2$; all three copies share that point at infinity; each two copies share one additional point $((0,0,0,0) = P_1 \cap P_2; (0,0,1,1) \in P_1 \cap P_3$; and $(1,1,0,0) \in P_2 \cap P_3)$. So by Alexander Duality, $\tilde{H}^i(Y) \cong \tilde{H}_{4-i}(X)$, which implies that $\tilde{H}^0(Y) \cong \mathbb{Z}$, $\tilde{H}^1(Y) \cong \mathbb{Z}^3$ and $\tilde{H}^2(Y) \cong \mathbb{Z}^3$.

[Additional remark: the cohomology ring is not too hard to get either, if we recognize that if only two of the planes are removed, what you get is homotopy equivalent to a torus. The cohomology ring ends up being isomorphic to $\mathbb{Z}[x_1, y_1, z_1]/(x^2 = y^2 = z^2 = xyz = 0)$.

4. (a) Show, using only the definition of tensor product, that tensoring with $\mathbb{Q}$ preserves exact sequences of abelian groups.

**Solution:** Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be exact. The map $id \otimes g : \mathbb{Q} \otimes B \to \mathbb{Q} \otimes C$ is surjective because any $x \otimes c$ will be the image of $x \otimes c$ where $g(c) = b$.

Next, note that in general $y \in \mathbb{Q} \otimes A$ could be $x_1 \otimes a_1 + \cdots + x_m \otimes a_m$. But $x_i \otimes a_i \sim \frac{1}{D} n_i \otimes a_i \sim \frac{1}{D} 1 \otimes n_i a_i$ where $D$ is the product of the denominators in the $x_i$. Thus any $y$ is of the form $x \otimes a$ for some $a$.

Now if $id \otimes f(x \otimes a) = x \otimes f(a)$ is zero, then this must be either equivalent to $y \otimes 0$ or $0 \otimes b$ for some $y$ or $b$. Because scalar multiplication in $\mathbb{Q}$ is injective we must have $x \otimes f(a) \sim \frac{1}{n} \otimes n f(a) = \frac{1}{n} \otimes 0$ for some $n$. But $0 = n f(a) = f(na)$. Because $f$ is injective, this means $na = 0$, which means $x \otimes a \sim \frac{1}{n} \otimes na = 0$, which implies that $id \otimes f$ is injective.

(b) Use the fiber sequences $SU(n-1) \to SU(n) \to S^{2n-1}$ (obtained by letting $SU(n)$ act on $S^{2n-1}$ as the unit sphere in $\mathbb{C}^n$, which has stabilizer $SU(n-1)$) to compute $\pi_*(SU(n)) \otimes \mathbb{Q}$ for $n \geq 2$. You may assume that $\pi_*(S^{2n-1}) \otimes \mathbb{Q}$ is $\mathbb{Q}$ if $* = 2n - 1$ and zero otherwise.

**Solution:** By the previous part and the fact that fiber sequences give rise to long exact sequences in homotopy, fiber sequences also give rise to long exact sequences in $\pi_* \otimes \mathbb{Q}$. In these long exact sequences, all maps are isomorphisms, so we have $\pi_i(SU(n)) \otimes \mathbb{Q} \cong \mathbb{Q}$ if $i$ is odd with $3 \leq i \leq 2n - 1$ and is zero otherwise.
5. Explicitly define Thom forms which generate $H^1(\mathbb{R}^3 - X)$ where $X$ is the union of the coordinate axes, proving that they span. You may assume the existence of real-valued functions with properties you specify.

**Solution:** First, by Alexander duality, $H^1(\mathbb{R}^3 - X) \cong H_1(\hat{X})$, where $\hat{X}$ is the one-point compactification of $X$. In this case, this one point compactification has six one cells, each joining two zero-cells. Contracting one of those edges to a point yields $\vee_5 S^1$, so $H_1 \cong \mathbb{R}^5$. If we consider $H_1(\mathbb{R}^3 - X)$, we can take the fundamental classes of circles such as $C_{z>0} = \{x^2 + y^2 = 1; z = 1\}$, $C_{y<0} = \{x^2 + z^2 = 1; y = -1\}$, etc to get a set of six classes. That these are linearly dependent can be seen as they are the boundary of a punctured $S^2$ centered at the origin. We use these classes to pair with the Thom forms.

Let $e(\theta)$ be a function on $S^1$ with support within $\varepsilon$ of $\theta = 0$ such that $\int_{S^1} f(\theta) \cos(\theta) d\theta = 1$. Then consider for example the one-form $\omega_{x+y^+} = e(\theta)e(\phi)dz$, where $\theta$ is the angle between a point and the positive quadrant of the $xy$-plane, measured from the $x$-axis and $\phi$ is the angle between a point and the $xy$-plane measured from the $y$-axis. By construction, $\omega_{x+y^+}$ is supported on a tubular neighborhood of the proper submanifold $A_{x+y^+} = \{z = 0; x > 0; y > 0\}$. Moreover, its integral over any segment from $(x, y, z)$ to $(x, y, -z)$ is one, for $z$ sufficiently large. So $\omega_{x+y^+}$ is a Thom form for $A_{x+y^+}$. Similarly define $A_{y+z^-}$ as $e(\theta)e(\phi)dx$, where $\theta$ now measures the angle from the $y > 0$ part of the $yz$-plane and $\phi$ the angle from the $z < 0$ part of that plane, and so forth for a total of twelve Thom forms.

If we consider $\omega_{x+y^+}$, etc. then we calculate their values on the circles listed above by counting intersections (with signs) with their corresponding submanifolds $A_{x+y^+}$ etc.

<table>
<thead>
<tr>
<th>$A_{x+y^+}$</th>
<th>$C_{x&gt;0}$</th>
<th>$C_{x&lt;0}$</th>
<th>$C_{y&gt;0}$</th>
<th>$C_{y&lt;0}$</th>
<th>$C_{z&gt;0}$</th>
<th>$C_{z&lt;0}$</th>
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</thead>
<tbody>
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</table>

These five rows are linearly independent, so the corresponding Thom forms span $H^1$.

6. (a) Let $M(\mathbb{Z}/n, 1)$ denote the Moore space obtained by attaching $D^2$ to $S^1$ using the map from the boundary of $D^2$ to $S^1$ sending $z$ to $z^n$. Explicitly construct a map $M(\mathbb{Z}/2^i, 1) \to M(\mathbb{Z}/2^{i+1}, 1)$ which on $H_1$ gives the standard inclusion of $\mathbb{Z}/2^i$ in $\mathbb{Z}/2^{i+1}$.

**Solution:** We map $M(\mathbb{Z}/2^i, 1) \to M(\mathbb{Z}/2^{i+1}, 1)$ by the identity map in the interior of $D^2$. This extends to a continuous map $f$ of the entire Moore space because on the boundary $S^1$ in $M(\mathbb{Z}/2^i, 1)$ we have that $x \sim y$ if and only if
\[ y/x \text{ is a multiple of } e^{2\pi i/2^i}, \text{ which means that it is a multiple of } e^{2\pi i/2^{i+1}} \text{ as well, so the identification is respected in } M(\mathbb{Z}/2^{i+1}, 1). \]

A generator of \( H_1(M(\mathbb{Z}/2^i, 1)) \) is represented by the arc from \( e^0 \) to \( e^{2\pi i/2^i} \) in the unit disk. Under the further identifications, this becomes two copies of the arc from \( e^0 \) to \( e^{2\pi i/2^{i+1}} \), so this generator maps to twice a generator of \( H_1(M(\mathbb{Z}/2^{i+1}, 1)) \), as required.

(b) Construct a space with \( H_1 \cong \mathbb{Z}/2^\infty = \mathbb{Z}[\frac{1}{2}]/\mathbb{Z} \).

**Solution:** Let \( Y = \bigcup_{i \in \mathbb{N}} M(\mathbb{Z}/2^i, 1) \times [i, i+1]/ \sim \) where \( x \times i \in M(\mathbb{Z}/2^{i-1}, 1) \times [i-1, i] \) is identified with \( f(x) \times i \in M(\mathbb{Z}/2^i, 1) \times [i, i+1] \). We claim that \( H_1(Y) \cong \mathbb{Z}/2^\infty \).

Consider the subspace \( Y_n \) which is the image of \( \bigcup_{0<i<n} M(\mathbb{Z}/2^i, 1) \times [i, i+1] \bigcup (M(\mathbb{Z}/2^n, 1) \times n) \). Then \( Y_n \) is a good subspace of \( Y_{n+1} \), considering the neighborhood \( \bigcup_{0<i<n} M(\mathbb{Z}/2^i, 1) \times [i, i+1] \bigcup (M(\mathbb{Z}/2^n, 1) \times [n, n+\varepsilon]) \) which clearly deformation retracts onto \( Y_n \). Moreover, \( Y_n \) itself deformation retracts onto \( M(\mathbb{Z}/2^n, 1) \times n \) by composing the retractions of \( M(\mathbb{Z}/2^i, 1) \times [i, i+1] \) onto \( M(\mathbb{Z}/2^i, 1) \times (i+1) \), starting with \( i = 0 \).

The homology of a union of subspaces, each good in the next, is the direct limit of their homology. By construction on \( H_1 \) we have the direct limit of \( \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/8 \to \cdots \), where each map is the standard inclusion. This direct limit is isomorphic to \( \mathbb{Z}/2^\infty \cong \mathbb{Z}[\frac{1}{2}]/\mathbb{Z} \), which admits a map from this directed system by sending the generator of \( \mathbb{Z}/2^i \) to \( \frac{1}{2^i} \). Since each such map is injective and since mapping the product of \( \mathbb{Z}/2^i \) to \( \mathbb{Z}[\frac{1}{2}]/\mathbb{Z} \) by the sum of these maps is surjective, we indeed have the limit.

7. Let \( D \) be a category whose collection of objects forms a set, and let \( C \) be a category in which all colimits exist. Let \( \text{Fun}(D, C) \) be the category whose objects are functors from \( D \) to \( C \) and whose morphisms are natural transformations.

(a) Complete the following definition with a single equality. Let \( F \) and \( G \) be two functors from \( D \) to \( C \), and \( \eta : F \to G \) a natural transformation. Let \( i \in D \) and let \( f_i \) denote the canonical map from \( F(i) \) to \( \text{colim} \ F \), the colimit of \( F \) (similarly, let \( g_i : G(i) \to \text{colim} \ G \)). Then the map \( \eta_* : \text{colim} \ F \to \text{colim} \ G \) induced by \( \eta \) is the unique map which for all \( i \) satisfies:

**Solution:** \( \eta_* \circ f_i = g_i \circ \eta(i) \).

(b) Explain why the map \( \eta_* \) from part (a) exists, and then show that with this definition taking \( F \) to the colimit of \( F \) defines a functor from \( \text{Fun}(D, C) \) to \( C \).
Solution: For each $i$, $\eta(i) : F(i) \rightarrow G(i)$ can be composed with the universal map $g(i) : G(i) \rightarrow \text{colim } G$. Because $\eta$ is a natural transformation and the universal maps $G(i) \rightarrow \text{colim } G$ commute with the structure maps of $G$, these $g(i) \circ \eta(i) : F(i) \rightarrow \text{colim } G$ commute with the structure maps of $F$. Thus by the definition of colimit, we get a map $\text{colim } F \rightarrow \text{colim } G$, which we define to be $\eta_*$, satisfying the condition above.

If $id$ is the identity natural transformation, then $id_* \circ f(i) = f(i) \circ id(i) = f(i)$. So $id_*$ must be the identity map, since that clearly does satisfy this condition.

If $H$ is another functor and $\tau : G \rightarrow H$ a natural transformation, we consider $\tau_* \circ \eta_*$. By definition, $\eta_* \circ f(i) = g(i) \circ \eta(i)$ and $\tau_* \circ g(i) = h(i) \circ \tau(i)$ for all $i$. Thus $(\tau_* \circ \eta_*) \circ f(i) = h(i) \circ (\tau(i) \circ \eta(i))$, so $\tau_* \circ \eta_*$ satisfies the defining property of $(\tau \circ \eta)_*$.

8. Show that the unit tangent bundle of $S^2$ is homeomorphic to $\mathbb{R}P^3$. Use this fact (whether you can prove it or not) to show that there are no non-vanishing vector fields on $S^2$.

Solution: The unit tangent bundle of $S^2$ can be identified with pairs $(u,v)$ where $u$ is a unit vector in $\mathbb{R}^3$ and $v$ is a unit vector in $u^\perp$. By then considering the matrix with columns $u,v,u \times v$ we get an element of $SO(3)$. This assignment is a homeomorphism, clearly continuous and the inverse just takes a matrix to the unit tangent vector defined by its first two columns.

On the other hand, we can also get a homeomorphism of $\mathbb{R}P^3$ with $SO(3)$. Consider the unit ball in $\mathbb{R}^3$ and send a point $x$ to the rotation of $\pi x$ radians around the unit vector $\pi x / \|x\|$ (when $x \neq 0$; if $x = 0$ send it to the identity matrix). Antipodal points on the boundary map to the same rotation (of $\pi = -\pi$ radians), so we get a well-defined map to $SO(3)$. Because any element of $SO(3)$ is a rotation about some axis, we get an inverse map, showing that this is a homeomorphism. Thus $UTS^2 \cong SO(3) \cong \mathbb{R}P^3$.

Next, a non-vanishing vector field on $S^2$ would by taking the corresponding unit vector field (dividing each vector by its length) yield map $s$ such that the composite $S^2 \xrightarrow{s} UTS^2 \xrightarrow{p} S^2$ is the identity. Here $p$ is the projection which sends a tangent vector to the point to which it is tangent. But if we apply $H_2$ we get the groups $Z \rightarrow Z/2 \rightarrow Z$. Since homology is functorial, the composite of these maps must be the identity, which is not possible.

9. Is there a space whose integral cohomology groups are given by $H^0 \cong \mathbb{Z}$, $H^1 \cong 0$, $H^2 \cong \mathbb{Z}/2$, $H^3 \cong \mathbb{Z}/3$ and $H^4 \cong \mathbb{Z}/6$? Justify your answer. Repeat the question where $H^*(-; \mathbb{Z})$ is replaced by $H^*(-; \mathbb{Z}/6)$. 
Solution: Yes, there is. Consider $X = M(\mathbb{Z}/2, 1) \lor M(\mathbb{Z}/3, 2) \lor M(\mathbb{Z}_6, 3)$. By definition, and the fact that the reduced homology of a wedge is the direct sum of the homology of the wedge summands, we have $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}/2$, $H_2 \cong \mathbb{Z}/3$, $H_3 \cong \mathbb{Z}/6$. We use the Universal Coefficient Theorem to compute homology. $\text{Hom}(\mathbb{Z}/n, \mathbb{Z}) = 0$ but $\text{Ext}^1(\mathbb{Z}/n, \mathbb{Z}) \cong \mathbb{Z}/n$. So the cohomology of $X$ is as desired.

There is not a space as with $H^*(X; \mathbb{Z}/6)$ as given. By the Universal Coefficient Theorem, if $H^2(X) \cong \text{Hom}(H_2(X), \mathbb{Z}/6) \oplus \text{Ext}^1(H_1(X), \mathbb{Z}/6)$. If this were $\mathbb{Z}/2$ then one and only one of $H_1(X)$ and $H_2(X)$ would have two-torsion but no torsion of order three. But in either case you would also get a summand of $\mathbb{Z}/2$ either in $H^1(X; \mathbb{Z}/6)$ or $H^3(X; \mathbb{Z}/6)$ respectively. Since this summand is not present, such cohomology groups are not possible.