Topology Qualifying Exam
Winter 2009

Name: ______________________________

1. Let QP^2 be the quaternionic projective space, which has cohomology with an additive basis of $1, y_4, {y_4}^2$. Show that any map from $X = QP^2 \times \mathbb{C}P^2$ to itself must have a fixed point.

2. Compute the homology groups of the manifold M obtained from $S^1 \times S^2$ by identifying $(x \times y) \sim (-x \times -y)$, where by $-x$ and $-y$ we mean the antipodal points to x and y respectively. [Hint: this is feasible with cellular homology; start with the cell structures on spheres with two cells in each dimension.] Verify Poincaré duality for $H_*(M; \mathbb{Z}/2)$.

3. (a) Consider three copies of S^2, which we denote $S^2_{(i)}$, with three specified points on each, namely $a_{(i)}$, $b_{(i)}$ and $c_{(i)} \in S^2_{(i)}$. Compute the homology groups of X, which is the union of these S^2's with identifications $a_{(1)} \sim a_{(2)} \sim a_{(3)}$ and $b_{(1)} \sim c_{(2)}$, $b_{(2)} \sim c_{(3)}$ and $b_{(3)} \sim c_{(1)}$.

   (b) Compute the cohomology groups of Y which is the complement in $\mathbb{R}^4$ of the planes $P_1 = \{(x,y,z,w)|x = y = 0\}$, $P_2 = \{(x,y,z,w)|z = w = 0\}$ and $P_3 = \{(x,y,z,w)|x + z = 1 = y + w\}$.

4. (a) Show, using only the definition of tensor product, that tensoring with $\mathbb{Q}$ preserves exact sequences of abelian groups.

   (b) Use the fiber sequences $SU(n-1) \to SU(n) \to S^{2n-1}$ (obtained by letting $SU(n)$ act on S^{2n-1} as the unit sphere in $\mathbb{C}^n$, which has stabilizer $SU(n-1)$) to compute $\pi_*(SU(n)) \otimes \mathbb{Q}$ for $n \geq 2$. You may assume that $\pi_*(S^{2n-1}) \otimes \mathbb{Q}$ is $\mathbb{Q}$ if $* = 2n-1$ and zero otherwise.

5. Explicitly define Thom forms which generate $H^1(\mathbb{R}^3 - X)$ where X is the union of the coordinate axes, proving that they span. You may assume the existence of real-valued functions with properties you specify.

6. (a) Let $M(\mathbb{Z}/n, 1)$ denote the Moore space obtained by attaching D^2 to S^1 using the map from the boundary of D^2 to S^1 sending z to z^n. Explicitly construct a map $M(\mathbb{Z}/2^i, 1) \to M(\mathbb{Z}/2^{i+1}, 1)$ which on H_1 gives the standard inclusion of $\mathbb{Z}/2^i$ in $\mathbb{Z}/2^{i+1}$.

   (b) Construct a space with $H_1 \cong \mathbb{Z}/2^\infty = \mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$.

7. Let $\mathcal{D}$ be a partially ordered set, viewed as a category with either one or no morphism between any two objects, and let $\mathcal{C}$ be a category in which all colimits exist. Let $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ be the category whose objects are functors from $\mathcal{D}$ to $\mathcal{C}$ and whose morphisms are natural transformations.

(a) Complete the following definition with a single equality. Let F and G be two functors from $\mathcal{D}$ to $\mathcal{C}$, and $\eta : F \longrightarrow G$ a natural transformation. Let $i \in \mathcal{D}$ and let f_i denote the canonical map from $F(i)$ to colim F, the colimit of F (similarly, let $g_i : G(i) \longrightarrow \text{colim } G$). Then the map $\eta_* : \text{colim } F \longrightarrow \text{colim } G$ induced by η is the unique map which for all i satisfies:

(b) Explain why the map η_* from part (a) exists, and then show that with this definition taking F to the colimit of F defines a functor from $\text{Fun}(\mathcal{D}, \mathcal{C})$ to $\mathcal{C}$.

8. Show that the unit tangent bundle of S^2 is homeomorphic to $\mathbb{R}P^3$. Use this fact (whether you can prove it or not) to show that there are no non-vanishing vector fields on S^2.

9. Is there a space whose integral cohomology groups are given by $H^0 \cong \mathbb{Z}$, $H^1 \cong 0$, $H^2 \cong \mathbb{Z}/2$, $H^3 \cong \mathbb{Z}/3$ and $H^4 \cong \mathbb{Z}/6$? Justify your answer. Repeat the question where $H^*(-;\mathbb{Z})$ is replaced by $H^*(-;\mathbb{Z}/6)$.

Topology Qualifying Exam
Winter 2009

Solutions

Name: ____________________

1. Let QP^2 be the quaternionic projective space, which has cohomology with an additive basis of $1, y_4, {y_4}^2$. Show that any map from $X = QP^2 \times \mathbb{C}P^2$ to itself must have a fixed point.

Solution: We use the Lefshetz theorem using $H^*(-;\mathbb{Z}/2)$. If we let x_2 be a generator of $H^*(\mathbb{C}P^2)$ then we have by the Künneth theorem that x and y together generate $H^*(X)$, since the cohomology of QP^2 and $\mathbb{C}P^2$ are torsion-free. Also because they are torsion free, we can use their mod-2 reductions as generators for $H^*(X;\mathbb{Z}/2)$. We see that under some map $f : X \to X$, f_* sends x to ax for some a. On the other hand y maps to $by + cx^2$ for some b or c (in fact, one of these must be zero, but we will not need that). Then the trace in each dimension is calculated as follows:

dim	0	2	4	6	8	10	12
tr	1	a	a^2+b	ab	b^2+a^2b	ab^2	a^2b^2.

Because each of these dimensions is even, we add these traces to get the Lefshetz number. Working modulo two, we use the identity $x^2 = x$ repeatedly to see that the Lefshetz number is one, guaranteeing a fixed point.

2. Compute the homology groups of the manifold M obtained from $S^1 \times S^2$ by identifying $(x \times y) \sim (-x \times -y)$, where by $-x$ and $-y$ we mean the antipodal points to x and y respectively. [Hint: this is feasible with cellular homology; start with the cell structures on spheres with two cells in each dimension.] Verify Poincaré duality for $H_*(M;\mathbb{Z}/2)$.

Solution: Put a cell structure on S^1 with two zero cells called $0_+ = (0,1) \in \mathbb{R}^2$ and $0_- = (0,-1) \in \mathbb{R}^2$ and two one-cells $1_+ = \{(x,y)|x > 0\}$ and $1_- = \{(x,y)|x < 0\}$ [it would suffice to draw a picture]. Put a similar cell structure on S^2. Then M has a cell structure with

3-cells	$1_+ \times 2_+ \sim 1_- \times 2_-$, $1_+ \times 2_- \sim 1_- \times 2_+$
2-cells	$1_+ \times 1_+$, $1_+ \times 1_-$, $0_+ \times 2_+$, $0_+ \times 2_-$
1-cells	$1_+ \times 0_+ \sim 1_- \times 0_-, 1_+ \times 0_-, 0_+ \times 1_+, 0_+ \times 1_-$
0-cells	$0_+ \times 0_+, 0_+ \times 0_-$

If we choose the standard orientations on our original cell structures such that in those cellular chain complexes, for example, $d(2_+) = 1_+ + 1_- = d(2_-)$ while $d(1_+) = 0_+ - 0_- = -d(1_-)$, then the identifications such as $1_+ \times 1_- \sim 1_- \times 1_+$ are orientation preserving. We thus may freely use the Leibniz rule to compute boundaries in the cellular chain complex of M, so for example

$$d(1_+ \times 2_+) = 0_+ \times 2_+ - 0_- \times 2_+ - 1_+ \times 1_+ - 1_+ \times 1_-.$$

We omit the rest of these standard calculations from our solution. The most complicated boundary homomorphism is from $C_2^{CW}(M)$ to $C_1^{CW}(M)$, which when using bases as listed above is represented by the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}.$$

We know $H_0(M) \cong \mathbb{Z}$ because it is connected.

For H_1, we see cycles spanned by $1_+ \times 0_+ + 1_- \times 0_+$, $1_+ \times 0_+ - 0_+ \times 1_+$ and $0_+ \times 1_+ + 0_+ \times 1_-$. The first cycle is d of $2_+ \times 0_+$ (and the additive inverse of d of $2_- \times 0_+$) and the difference of the first and third is d of $1_+ \times 1_+$. The image of d is generated by these two cases. Thus, $H_1 \cong \mathbb{Z}$.

For H_2 we see cycles spanned by $a = 1_+ \times 1_+ + 1_+ \times 1_-$ and $b = 0_+ \times 2_+ - 0_+ \times 2_-$. But $d(1_+ \times 2_+) = b - a$ and $d(1_+ \times 2_-) = -b - a$. Thus $H_2 \cong \mathbb{Z}/2$,

Because d is injective on $C_3^{CW}(M)$, $H_3 = 0$.

With $\mathbb{Z}/2$ coefficients we get $H_3 \cong H_0 \cong \mathbb{Z}/2 \cong H_2 \cong H_1$, satisfying Poincaré duality.

3. (a) Consider three copies of S^2, which we denote $S^2_{(i)}$, with three specified points on each, namely $a_{(i)}$, $b_{(i)}$ and $c_{(i)} \in S^2_{(i)}$. Compute the homology groups of X, which is the union of these S^2's with identifications $a_{(1)} \sim a_{(2)} \sim a_{(3)}$ and $b_{(1)} \sim c_{(2)}$, $b_{(2)} \sim c_{(3)}$ and $b_{(3)} \sim c_{(1)}$.

Solution: After the first identification we have $\bigvee_3 S^2$, which has reduced homology of rank three in degree two and zero otherwise. When the identification $b_{(1)} \sim c_{(2)}$ is made, we consider the corresponding sequence $S^0 \hookrightarrow \vee_3 S^2 \to Z = \bigvee_3 S^2/(b_{(1)} \sim c_{(2)})$. The interesting part of the long exact sequence in reduced homology reads

$$\cdots \to \tilde{H}_1(\bigvee_3 S^2) = 0 \to \tilde{H}_1(Z) \to \tilde{H}_0(S^0) \cong \mathbb{Z} \to \tilde{H}_0(\bigvee_3 S^2) = 0.$$

So we get $H_0(Z) \cong H_1(Z) \cong \mathbb{Z}$ and $H_2(Z) \cong \mathbb{Z}^3$. The analysis goes through in exactly the same manner for the further two identifications, yielding $H_0(X) \cong \mathbb{Z}$ and $H_1(X) \cong H_2(X) \cong \mathbb{Z}^3$.

(b) Compute the cohomology groups of Y which is the complement in $\mathbb{R}^4$ of the planes $P_1 = \{(x,y,z,w)|x = y = 0\}$, $P_2 = \{(x,y,z,w)|z = w = 0\}$ and $P_3 = \{(x,y,z,w)|x+z = 1 = y+w\}$.

Solution: Consider S^4 as the one-point compactification of $\mathbb{R}^4$, so that the complement of the one-point compactification of the planes in S^4 is homeomorphic to the complement of the planes themselves in $\mathbb{R}^4$. But the one-point compactification of the planes is homeomorphic to X from part (a): each plane with one point at infinity yields a copy of S^2; all three copies share that point at infinity; each two copies share one additional point ($(0,0,0,0) = P_1 \cap P_2$; $(0,0,1,1) \in P_1 \cap P_3$; and $(1,1,0,0) \in P_2 \cap P_3$). So by Alexander Duality, $\tilde{H}^i(Y) \cong \tilde{H}_{4-i-i}(X)$, which implies that $H^0(Y) \cong \mathbb{Z}$, $H^1(Y) \cong \mathbb{Z}^3$ and $H^2(Y) \cong \mathbb{Z}^3$.

[Additional remark: the cohomology ring is not too hard to get either, if we recognize that if only two of the planes are removed, what you get is homotopy equivalent to a torus. The cohomology ring ends up being isomorphic to $\mathbb{Z}[x_1, y_1, z_1]/(x^2 = y^2 = z^2 = xzy = 0)$.]

4. (a) Show, using only the definition of tensor product, that tensoring with $\mathbb{Q}$ preserves exact sequences of abelian groups.

Solution: Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be exact. The map $id \otimes g : \mathbb{Q} \otimes B \to \mathbb{Q} \otimes C$ is surjective because any $x \otimes c$ will be the image of $x \otimes c$ where $g(c) = b$.
Next, note that in general $y \in \mathbb{Q} \otimes A$ could be $x_1 \otimes a_1 + \cdots + x_m \otimes a_m$. But $x_i \otimes a_i \sim \frac{1}{D} n_i \otimes a_i \sim \frac{1}{D} 1 \otimes n_i a_i$ where D is the product of the denominators in the x_i. Thus any y is of the form $x \otimes a$ for some a.
Now if $id \otimes f(x \otimes a) = x \otimes f(a)$ is zero, then this must be either equivalent to $y \otimes 0$ or $0 \otimes b$ for some y or b. Because scalar multiplication in $\mathbb{Q}$ is injective we must have $x \otimes f(a) \sim \frac{x}{n} \otimes nf(a) = \frac{x}{n} \otimes 0$ for some n. But $0 = nf(a) = f(na)$. Because f is injective, this means $na = 0$, whcih means $x \otimes a \sim \frac{x}{n} \otimes na = 0$, which implies that $id \otimes f$ is injective.

(b) Use the fiber sequences $SU(n-1) \to SU(n) \to S^{2n-1}$ (obtained by letting $SU(n)$ act on S^{2n-1} as the unit sphere in $\mathbb{C}^n$, which has stabilizer $SU(n-1)$) to compute $\pi_*(SU(n)) \otimes \mathbb{Q}$ for $n \geq 2$. You may assume that $\pi_*(S^{2n-1}) \otimes \mathbb{Q}$ is $\mathbb{Q}$ if $* = 2n-1$ and zero otherwise.

Solution: By the previous part and the fact that fiber sequences give rise to long exact sequences in homotopy, fiber sequences also give rise to long exact sequences in $\pi_* \otimes \mathbb{Q}$. In these long exact sequences, all maps are isomorphisms, so we have $\pi_i(SU(n)) \otimes \mathbb{Q} \cong \mathbb{Q}$ if i is odd with $3 \leq i \leq 2n-1$ and is zero otherwise.

5. Explicitly define Thom forms which generate $H^1(\mathbb{R}^3 - X)$ where X is the union of the coordinate axes, proving that they span. You may assume the existence of real-valued functions with properties you specify.

Solution: First, by Alexander duality, $H^1(\mathbb{R}^3 - X) \cong H_1(\hat{X})$, where $\hat{X}$ is the one-point compactification of X. In this case, this one point compactifcation has six one cells, each joining two zero-cells. Contracting one of those edges to a point yields $\bigvee_5 S^1$, so $H_1 \cong \mathbb{R}^5$. If we consider $H_1(\mathbb{R}^3 - X)$, we can take the fundamental classes of circles such as $C_{z>0} = \{x^2 + y^2 = 1; z = 1\}$, $C_{y<0} = \{x^2 + z^2 = 1; y = -1\}$, etc to get a set of six classes. That these are linearly dependent can be seen as they are the boundary of a punctured S^2 centered at the origin. We use these classes to pair with the Thom forms.

Let $e(\theta)$ be a function on S^1 with support within ε of $\theta = 0$ such that $\int_{S^1} f(\theta)\cos(\theta)d\theta =$ 1. Then consider for example the one-form $\omega_{x+y+} = e(\theta)e(\phi)dz$, where θ is the angle between a point and the positive quadrant of the xy-plane, measured from the x-axis and ϕ is the angle between a point and the xy-plane measured from the y-axis. By construction, ω_{x+y+} is supported on a tubular neighborhood of the proper submanifold $A_{x+y+} = \{z = 0; x > 0; y > 0\}$. Moreover, its integral over any segment from (x, y, z) to $(x, y, -z)$ is one, for z sufficiently large. So ω_{x+y+} is a Thom form for A_{x+y+}. Similarly define A_{y+z-} as $e(\theta)e(\phi)dx$, where θ now measures the angle from the $y > 0$ part of the yz-plane and ϕ the angle from the $z < 0$ part of that plane, and so forth for a total of twelve Thom forms.

If we consider ω_{x+y+}, etc. then we calculate their values on the circles listed above by counting intersections (with signs) with their corresponding submanifolds A_{x+y+} etc.

We get intersection data including:

	$C_{x>0}$	$C_{x<0}$	$C_{y>0}$	$C_{y<0}$	$C_{z>0}$	$C_{z<0}$
A_{x+y+}	±1	0	±1	0	0	0
A_{x+y-}	±1	0	0	±1	0	0
A_{x-y-}	0	±1	0	±1	0	0
A_{x+z+}	±1	0	0	0	±1	0
A_{x-z-}	0	±1	0	0	0	±1

These five rows are linearly independent, so the corresponding Thom forms span H^1.

6. (a) Let $M(\mathbb{Z}/n, 1)$ denote the Moore space obtained by attaching D^2 to S^1 using the map from the boundary of D^2 to S^1 sending z to z^n. Explicitly construct a map $M(\mathbb{Z}/2^i, 1) \to M(\mathbb{Z}/2^{i+1}, 1)$ which on H_1 gives the standard inclusion of $\mathbb{Z}/2^i$ in $\mathbb{Z}/2^{i+1}$.

Solution: We map $M(\mathbb{Z}/2^i, 1)$ to $M(\mathbb{Z}/2^{i+1}, 1)$ by the identity map in the interior of D^2. This extends to a continuous map f of the entire Moore space because on the boundary S^1 in $M(\mathbb{Z}/2^i, 1)$ we have that $x \sim y$ if and only if

y/x is a multiple of $e^{2\pi i/2^i}$, which means that it is a multiple of $e^{2\pi i/2^{i+1}}$ as well, so the identification is respected in $M(\mathbb{Z}/2^{i+1},1)$.

A generator of $H_1(M(\mathbb{Z}/2^i,1))$ is represented by the arc from e^0 to $e^{2\pi i/2^i}$ in the unit disk. Under the further identifications, this becomes two copies of the arc from e^0 to $e^{2\pi i/2^{i+1}}$, so this generator maps to twice a generator of $H_1(M(\mathbb{Z}/2^{i+1},1)$, as required.

(b) Construct a space with $H_1 \cong \mathbb{Z}/2^\infty = \mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$.

Solution: Let $Y = \bigcup_{i\in\mathbb{N}} M(\mathbb{Z}/2^i,1))\times[i,i+1]/\sim$ where $x\times i \in M(\mathbb{Z}/2^{i-1},1)\times[i-1,i]$ is identified with $f(x)\times i \in M(\mathbb{Z}/2^i,1)\times[i,i+1]$. We claim that $H_1(Y)\cong\mathbb{Z}/2^\infty$.

Consider the subspace Y_n which is the image of $\bigcup_{0<i<n} M(\mathbb{Z}/2^i,1))\times[i,i+1]\bigcup(M(\mathbb{Z}/2^n,1)\times n)$. Then Y_n is a good subspace of Y_{n+1}, considering the neighborhood $\bigcup_{0<i<n} M(\mathbb{Z}/2^i,1))\times[i,i+1]\bigcup(M(\mathbb{Z}/2^n,1)\times[n,n+\varepsilon))$ which clearly deformation retracts onto Y_n. Moreover, Y_n itself deformation retracts onto $M(\mathbb{Z}/2^n,1)\times n$ by composing the retractions of $M(\mathbb{Z}/2^i,1))\times[i,i+1]$ onto $M(\mathbb{Z}/2^i,1))\times(i+1)$, starting with $i=0$.

The homology of a union of subspaces, each good in the next, is the direct limit of their homology. By construction on H_1 we have the direct limit of $\mathbb{Z}/2\to\mathbb{Z}/4\to\mathbb{Z}/8\to\cdots$, where each map is the standard inclusion. This direct limit is isomorphic to $\mathbb{Z}/2^\infty\cong\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$, which admits a map from this directed system by sending the generator of $\mathbb{Z}/2^i$ to $\frac{1}{2^i}$. Since each such map is injective and since mapping the product of $\mathbb{Z}/2^i$ to $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$ by the sum of these maps is surjective, we indeed have the limit.

7. Let $\mathcal{D}$ be a category whose collection of objects forms a set, and let $\mathcal{C}$ be a category in which all colimits exist. Let $\mathrm{Fun}(\mathcal{D},\mathcal{C})$ be the category whose objects are functors from $\mathcal{D}$ to $\mathcal{C}$ and whose morphisms are natural transformations.

(a) Complete the following definition with a single equality. Let F and G be two functors from $\mathcal{D}$ to $\mathcal{C}$, and $\eta : F\to G$ a natural transformation. Let $i\in\mathcal{D}$ and let f_i denote the canonical map from $F(i)$ to colim F, the colimit of F (similarly, let $g_i : G(i)\to$ colim G). Then the map $\eta_* :$ colim $F\to$ colim G induced by η is the unique map which for all i satisfies:

Solution: $\eta_*\circ f_i = g_i\circ\eta(i)$.

(b) Explain why the map η_* from part (a) exists, and then show that with this definition taking F to the colimit of F defines a functor from $\mathrm{Fun}(\mathcal{D},\mathcal{C})$ to $\mathcal{C}$.

Solution: For each i, $\eta(i) : F(i) \to G(i)$ can be composed with the universal map $g(i) : G(i) \to \operatorname{colim} G$. Because η is a natural transformation and the universal maps $G(i) \to \operatorname{colim} G$ commute with the structure maps of G, these $g(i) \circ \eta(i) : F(i) \to \operatorname{colim} G$ commute with the structure maps of F. Thus by the definition of colimit, we get a map $\operatorname{colim} F \to \operatorname{colim} G$, which we define to be η_*, satisfying the condition above.

If id is the identity natural transformation, then id_* uniquely satisfies $id_* \circ f(i) = f(i) \circ id(i) = f(i)$. So id_* must be the identity map, since that clearly does satisfy this condition.

If H is another functor and $\tau : G \to H$ a natural transformation, we consider $\tau_* \circ \eta_*$. By definition, $\eta_* \circ f(i) = g(i) \circ \eta(i)$ and $\tau_* \circ g(i) = h(i) \circ \tau(i)$ for all i. Thus $(\tau_* \circ \eta_*) \circ f(i) = h(i) \circ (\tau(i) \circ \eta(i))$, so $\tau_* \circ \eta_*$ satisfies the defining property of $(\tau \circ \eta)_*$.

8. Show that the unit tangent bundle of S^2 is homeomorphic to $\mathbb{R}P^3$. Use this fact (whether you can prove it or not) to show that there are no non-vanishing vector fields on S^2.

Solution: The unit tangent bundle of S^2 can be identified with pairs (u, v) where u is a unit vector in $\mathbb{R}^3$ and v is a unit vector in $u^\perp$. By then considering the matrix with columns $u, v, u \times v$ we get an element of $SO(3)$. This assignment is a homeomorphism, clearly continuous and the inverse just takes a matrix to the unit tangent vector defined by its first two columns.

On the other hand, we can also get a homeomorphism of $\mathbb{R}P^3$ with $SO(3)$. Consider the unit ball in $\mathbb{R}^3$ and send a point x to the rotation of πx radians around the unit vector $\frac{x}{||x||}$ (when $x \neq 0$; if $x = 0$ send it to the identity matrix). Antipodal points on the boundary map to the same rotation (of $\pi = -\pi$ radians), so we get a well-defined map to $SO(3)$. Because any element of $SO(3)$ is a rotation about some axis, we get an inverse map, showing that this is a homeomorphism. Thus $UTS^2 \cong SO(3) \cong \mathbb{R}P^3$.

Next, a non-vanishing vector field on S^2 would by taking the corresponding unit vector field (dividing each vector by its length) yield map s such that the composite $S^2 \xrightarrow{s} UTS^2 \xrightarrow{p} S^2$ is the identity. Here p is the projection which sends a tangent vector to the point to which it is tangent. But if we apply H_2 we get the groups $\mathbb{Z} \to \mathbb{Z}/2 \to \mathbb{Z}$. Since homology is functorial, the composite of these maps must be the identity, which is not possible.

9. Is there a space whose integral cohomology groups are given by $H^0 \cong \mathbb{Z}$, $H^1 \cong 0$, $H^2 \cong \mathbb{Z}/2$, $H^3 \cong \mathbb{Z}/3$ and $H^4 \cong \mathbb{Z}/6$? Justify your answer. Repeat the question where $H^*(-;\mathbb{Z})$ is replaced by $H^*(-;\mathbb{Z}/6)$.

Solution: Yes, there is. Consider $X = M(\mathbb{Z}/2,1) \vee M(\mathbb{Z}/3,2) \vee M(\mathbb{Z}_6,3)$. By definition, and the fact that the reduced homology of a wedge is the direct sum of the homology of the wedge summands, we have $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}/2$, $H_2 \cong \mathbb{Z}/3$, $H_3 \cong \mathbb{Z}/6$. We use the Universal Coefficient Theorem to compute homology. $\mathrm{Hom}(\mathbb{Z}/\mathrm{n},\mathbb{Z}) = 0$ but $\mathrm{Ext}^1(\mathbb{Z}/n,\mathbb{Z}) \cong \mathbb{Z}/n$. So the cohomology of X is as desired.

There is not a space as with $H^*(X;\mathbb{Z}/6)$ as given. By the Universal Coefficient Theorem, if $H^2(X) \cong \mathrm{Hom}(H_2(X),\mathbb{Z}/6) \oplus \mathrm{Ext}^1(H_1(X),\mathbb{Z}/6)$. If this were $\mathbb{Z}/2$ then one and only one of $H_1(X)$ and $H_2(X)$ would have two-torsion but no torsion of order three. But in either case you would also get a summand of $\mathbb{Z}/2$ either in $H^1(X;\mathbb{Z}/6)$ or $H^3(X;\mathbb{Z}/6)$ respectively. Since this summand is not present, such cohomology groups are not possible.