Topology Qual, Winter 2008

(1) Let X be path connected, locally path connected, and locally simply connected with basepoint x_0.

   (a) Give an explicit construction of $\tilde{X}$, the universal cover of X. You should define the topology on this space, but you need not prove anything else about this construction (in particular you don't need to prove it *is* the universal cover of X).

   (b) Assuming the part above, if $H \subseteq \pi_1(X, x_0)$, prove there is a covering space $\tilde{X}$ with basepoint x_1 so that the image of $\pi_1(\tilde{X}, x_1)$ is H.

(2) Let X be a CW-complex, and A a contractible subcomplex. Prove that the quotient map $X \longrightarrow X/A$ is a homotopy equivalence.

(3) (a) Prove that there is an element of $\pi_{4n-1}(S^{2n})$ of Hopf invariant 2.

   (b) Prove that $\mathbf{Z}$ is a summand of $\pi_{4n-1}(S^{2n})$.

(4) Prove that if X and Y are CW-complexes,

$$X * Y \simeq \Sigma X \wedge Y.$$

(5) Recall that a path γ from x_0 to x_1 gives an isomorphism $h_\gamma : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$.

   Show $\pi_1(X, x_0)$ is abelian if and only if h_γ depends only on x_0, x_1 and not on γ.

(6) Use the method of acyclic models to show $\alpha \cup \beta = (-1)^{|\alpha||\beta|} \beta \cup \alpha$.

(7) Assume that if $f : S^{2n} \longrightarrow S^{2n}$ is continuous. Show there is an x so that $f(x) = \pm x$.

(8) (a) Prove that if $n \geq 1$, d an integer, there is a map $f : S^n \longrightarrow S^n$ of degree d.

   (b) Let G be an abelian group. Prove there is a CW complex $M(G, n)$ which has $\tilde{H}_*(M(G, n)) = G$ if $* = n$ and 0 else.

(9) Suppose X is a cell complex with $\tilde{H}_*(X) = 0$. Prove that ΣX is contractible.

(10) (a) Give examples of CW complexes X, Y where $H_*(X) = H_*(Y)$ but $X \not\simeq Y$.

   (b) Give examples of CW complexes X, Y where $\pi_*(X) = \pi_*(Y)$ but $X \not\simeq Y$.