Topology Qualifying Exam
Winter 2007

1. Let $p\colon E \to B$ be a covering space, and let $A \hookrightarrow X$ be an inclusion where A is nonempty and X is contractible. Assume one has a map $f\colon X \to B$, and a map $g\colon A \to E$ such that $p \circ g = f|_A$. Prove that there is a map $F\colon X \to E$ such that $F|_A = g$.

2. Let M be a compact, orientable n-manifold and let N be a compact, non-orientable n-manifold. Compute all the groups $H_i(M\#N)$, expressing them in terms of $H_*(M)$ and $H_*(N)$.

3. Let K be the Klein bottle, and recall that $\pi_1(K) = \langle a, b \,|\, aba = b\rangle$. Determine the universal covering space of K, perhaps by drawing a picture. Also, describe a pointed covering space $p\colon E \to K$ such that $p_*(\pi_1(E, e)) = \langle a^2, ab\rangle$. How many points are in each fiber of p?

4. Let Y be the space obtained by starting with S^3 and attaching a 4-cell via a map of degree $k > 0$: $Y = S^3 \cup_f e^4$ where $f\colon \partial(e^4) \to S^3$ has degree k. Compute the homology and cohomology groups of $\mathbb{R}P^4 \times Y$, in terms of k, in the two cases where k is even and k is odd.

5. (a) If $\alpha \in H^p(X)$ and $\beta \in H^q(X)$ are singular cochains, define $\alpha \cup \beta$.
   (b) Prove that if $X = \Sigma Y$ for some space Y, then $\alpha \cup \beta = 0$ provided $p, q > 0$.

6. Let M be a compact, non-orientable 3-manifold. Prove that $\operatorname{rank} H_1(M) = \operatorname{rank} H_2(M) + 1$. [Hint: Think also about the $\mathbb{Z}/2^e$-summands in the two groups.]

7. Someone claims that they can attach a 5-cell to $\mathbb{C}P^2$ to make a space X with $H_4(X) = \mathbb{Z}/3$. Prove that this is impossible by first showing that any map $S^4 \to \mathbb{C}P^2$ induces the zero map on $H_4(-)$.

8. Let X be the quotient space of $[-1,1] \times [-1,1] \times [-1,1]$ by the relations

$$(1, y, z) \sim (-1, -y, z), \qquad (x, 1, z) \sim (x, -1, z), \qquad \text{and} \qquad (x, y, 1) \sim (x, y, -1).$$

   Compute $H_i(X)$ for all i. It turns out that X is a manifold; decide whether it is orientable or not, and justify your answer.

9. (a) Compute $\pi_i(\mathbb{R}P^3 \times \mathbb{C}P^2)$ for $1 \le i \le 3$.
   (b) Let p be any point in $\mathbb{R}P^3 \times \mathbb{C}P^2$ and let $X = (\mathbb{R}P^3 \times \mathbb{C}P^2) - \{p\}$. Prove that $\pi_i(X) \to \pi_i(\mathbb{R}P^3 \times \mathbb{C}P^2)$ is an isomorphism for $1 \le i \le 3$. [You can assume p is in the top cell of $\mathbb{R}P^3 \times \mathbb{C}P^2$.]