1	2	3	4	5	6	7	8	9	10	
11	12	13	14	15	16	17	18	19	20	Total

QUALIFYING EXAM, Winter 2006

Algebraic Topology and Differential Geometry

NAME ____

(PRINT LAST AND THE FIRST NAME)

STUDENT NUMBER ____________________ SIGNATURE ____________________

Please do any 10 problems out of the following 20.

1. Prove the spaces $\mathbf{RP}^n \times S^k$ and $S^n \times \mathbf{RP}^k$ are homotopy equivalent if and only if $k = n$.
2. State and prove the Jordan-Brouwer Theorem.
3. Consider the map

$$g : S^{2n-2} \times S^3 \xrightarrow{\text{projection}} (S^{2n-2} \times S^3)/(S^{2n-2} \vee S^3) = S^{2n+1} \xrightarrow{\text{Hopf}} \mathbf{CP}^n.$$

Prove that g induces trivial homomorphism in homology and homotopy groups, however g is not homotopic to a constant map.

4. Define cup product in cohomology. Let $n > k$. Prove that there is no map $f : \mathbf{CP}^n \to \mathbf{CP}^k$ which induces nontrivial homomorphism

$$f_* : \pi_2\mathbf{CP}^n \to \pi_2\mathbf{CP}^k.$$

5. Let (K, L) be a CW-pair, Z be a compact Hausdorff topological space. Let $E = \mathcal{C}(K, Z)$, $B = \mathcal{C}(L, Z)$ be the spaces of continuous maps. A map $p : E \longrightarrow B$ be defined as $p(f : K \longrightarrow Z) = (f|_L : L \longrightarrow Z)$. Prove that the map $p : E \longrightarrow B$ is a Serre fiber bundle.
6. Compute cohomology ring $H^*(X; \mathbf{Z}/2)$ for $X := \mathbf{RP}^n \setminus \{x_0, x_1\}$, where $x_0 \neq x_1$ are any two points in $\mathbf{RP}^n$.
7. Use $\mathbf{Z}/2$-cohomology to prove that there is no continuous map

$$f : \mathbf{RP}^3 \times \mathbf{RP}^4 \to \mathbf{RP}^6$$

such that the restrictions

$$f|_{\mathbf{RP}^3 \times \{pt\}} : \mathbf{RP}^3 \times \{pt\} \to \mathbf{RP}^6, \qquad f|_{\{pt\} \times \mathbf{RP}^4} : \{pt\} \times \mathbf{RP}^4 \to \mathbf{RP}^6$$

are the standard inclusions.

8. Prove the isomorphism

$$\pi_{n+k}(S^{n+1} \vee S^{k+1}) \cong \pi_{n+k}(S^{n+1}) \oplus \pi_{n+k}(S^{k+1})$$

9. Let $n \geq 2$. Compute the homotopy groups $\pi_k(\mathbf{RP}^n)$ for $1 \leq k < n$.
10. Define the Hopf invariant $h(\alpha)$ of an element $\lambda \in \pi_{4q-1}(S^{2q})$. Compute the Hopf invariant of the generator of $\pi_3 S^2$.